# A potential hierarchy of properties

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# Introduction

Potentialism about sets—at least the version I'm going to be interested in today—is, in a slogan, the view that:

*set existence is a matter of co-existence*

A little more precisely, it's the view that:

*possible set existence is a matter of possible co-existence*

For example, since the people in this room all co-exist, set potentialism will tell us they could form a set. Similarly, there could be a set of the objects in our galaxy. And, there could be a set of all the objects whatsoever!

In general, no matter what things you have, they could form a set!

What is the relevant notion of possibility here? The sense in which any things **can** form a set?

There are different options, but for Linnebo and Studd, it's **interpretational**. I'm going to follow them.

Very roughly, "it's possible that $\phi$" means something like "it's possible to reinterpret the language so that $\phi$ is true".

So, the set potentialist thinks that no matter what possible interpretation $I$ of the language of set theory we consider, there could be an interpretation $I'$ extending $I$ with a set of any things in the domain of $I$.

Typically, some of these sets will be new. By the reasoning of Russell's paradox, for example, the set of non-self-membered sets according to $I$ will have to be new in $I'$.

Of course, we don't have to stop at $I'$. We can take any new things in the domain of $I'$ and find a further possible interpretation $I''$ in which they form a set. And so on.

In general, by iterating this procedure over any possible interpretation, we generate a rich universe of possible sets. Enough, given the right background assumptions, to satisfy the axioms of ZFC!

The goal of this talk is to investigate an analogous view for untyped properties (that is, properties as objects). It is, in a slogan, the view that:

*property existence is a matter of definition*

A little more precisely, it's the view that:

*possible property existence is a matter of possible definition*

For example, since we can define the property of being self-identical in the language we currently speak, property potentialism will tell us that there could be the property of being self-identical. More interesting, no matter what we actually mean by "property", property potentialism will tell us that there could be a property of being a property in that sense!

In general, no matter what definition we have, it could define a property.

So, the property potentialist thinks that no matter what possible interpretation $I$ we consider, there could be an interpretation $I'$ extending $I$ with the property of being $\phi$-interpreted-according-to-$I$.

Of course, we don't have to stop at $I'$. We can take any new definition in $I'$ and find a further possible interpretation in which it defines a property. And so on.

In general, by iterating this procedure over any possible interpretation, we can generate a rich universe of possible properties. As I will show, enough, given the right background assumptions, to satisfy the full comprehension schema for typed properties! (In a sense I'll articulate soon.)

# Plan

- I'll start by outlining two different responses to Russell's paradox. This will help to explain the core difference between set and property potentialism.
- I'll then present a formal theory for property potentialism and show that it's remarkably strong.
- Finally, I'll raise a general problem for the theory. This leads to a (in my opinion negative) conjecture about property potentialism.

# Ontological expansion

The reasoning of Russell's paradox tells us there's no one who makes salads for all and only those people who don't make salads for themselves. Formally, it tells us that

$$\exists x \forall y (Salad(x, y) \leftrightarrow \neg Salad(y, y))$$

is inconsistent.

There are two natural ways to "get around" this inconsistency.

The first is to take $\exists x$ to range over a broader domain than $\forall y$. I'll call this the **ontological expansion** strategy.

Here's an example: given any group of people $\mathcal{G}$, although it's not possible to find someone in $\mathcal{G}$ who makes salads for all and only the people in $\mathcal{G}$ who don't make salads for themselves, it is possible to find some such person **outside of** $\mathcal{G}$.

$$\exists x \forall y \in \mathcal{G}(Salad(x, y) \leftrightarrow \neg Salad(y, y))$$

Ontological expansion is the mechanism I take to underlie set potentialism.

We start with some possible interpretation $I$. We take any things from its domain that don't already form a set in $I$. We then extend $I$ to a new interpretation $I'$ with an object that has precisely those things as members according to $I'$.

Since sets are completely characterised by what members they have, that object plays the role of the corresponding set.

We can express the basic idea more precisely and more bloodlessly using a modal operator $\Diamond$ intended to capture the sense in which possible interpretations can be extended and backtracking operators $\uparrow$ and $\downarrow$ that allow us to index and access specific interpretational possibilities.

(Set Collapse$^\Diamond$) $$\uparrow \Diamond \exists s \forall y (y \in s \leftrightarrow \phi \wedge \downarrow Ey)$$

# Ideological expansion

A second way to avoid the inconsistency of:

$$\exists x \forall y (Salad(x, y) \leftrightarrow \neg Salad(y, y))$$

is to take the left-hand "salad" to have a broader meaning than the right-hand "salad". Here's an example.

We used to use "salad" in a narrower sense than we currently do. Now, a fruit salad is considered a salad, but it didn't use to be. Let $Salad_0$ denote the old use, and $Salad_1$ the new use. Then it's perfectly consistent that:

$$\exists x \forall y (Salad_1(x, y) \leftrightarrow \neg Salad_0(y, y))$$

I'll call this the **ideological expansion** strategy.

Ideological expansion is the mechanism I take to underlie the kind of property potentialism I'm interested in.

We start with some possible interpretation $I$. We take any definition available in $I$ that doesn't define a property in $I$. We then extend $I$ to a new interpretation $I'$ with an object that applies in any circumstance to the things satisfying the definition.

Since properties are completely characterised by their application conditions, that object plays the role of the corresponding property.

Again, we can formulate the idea more precisely and bloodlessly as follows.

$$\text{(Property Collapse}^{\Diamond}\text{)} \qquad \uparrow \Diamond \exists p \Box \forall y \Box (y \eta p \leftrightarrow \downarrow \phi)$$

where $y \eta p$ means that the property $p$ applies to $y$.

# The difference matters

In both cases, we're doing a bit of ontological expansion and a bit of ideological expansion. But the difference matters!

Because the newly introduced sets are outside the domains their members are from, we can have sets for any condition over those domains.

That's what Set Collapse$^\Diamond$ tells us.

Because the newly introduced properties have an application relation outside the application relation of the interpretation they are defined over, we can have properties that apply to absolutely all the things satisfying the definition.

That's what Property Collapse$^\Diamond$ tells us.

So, importantly, although there can't be a set of all possible objects, there can be a property applying to all possible objects.

# A formal theory for one version of property potentialism

I'm now going to outline a formal theory for property potentialism based on Property Collapse$^\Diamond$.

We start with an interpreted first-order language $\mathcal{L}$. For simplicity, I'll assume that $\mathcal{L}$ has just a relation symbol $R$ and a predicate symbol $N$. I'll call this the **base language**.

Question: What interpretation can $R$ and $N$ have?

Answer: Any possible interpretation whatsoever!

**PSA:** Are you interested in higher-order, plural, or modal languages? Ask me about them in the Q&A!

We now extend $\mathcal{L}$ with some property-theoretic resources: a relation symbol $\eta$ for property application and a predicate symbol $P$ for being a property.

To begin with, $\eta$ and $P$ are interpreted to be empty. $\eta$ relates nothing to nobody nowhere, and $P$ applies to nothing. As we move from possible interpretation to possible interpretation, their meanings will expand.

Let $\mathcal{L}_\eta$ be $\mathcal{L} + \{\eta, P\}$.

Next, we add some modal resources. First, we add an operator $\Diamond$ which is intended to express the sense in which possible interpretations can be extended with respect to the meanings of $\eta$ and $P$. Second, we add backtracking operators $\uparrow$ and $\downarrow$ that allow us to talk about given interpretational possibilities within the scope of other interpretational possibilities.

Let $\mathcal{L}_\eta^\Diamond$ be $\mathcal{L}_\eta + \{\Diamond, \uparrow, \downarrow\}$.

The modal logic for $\mathcal{L}^{\Diamond}$ is (almost) immediate, given the way we're thinking about $\Diamond$, $\uparrow$, $\downarrow$.

Over a standard positive free logic, I'll first adopt the principles of the minimal normal modal logic K for $\Diamond, \uparrow$, and $\downarrow$.

Since $\Diamond$ concerns **extensions** of a given interpretational possibility, I;ll also adopt:

(T) $$\phi \to \Diamond\phi$$

(4) $$\Diamond\Diamond\phi \to \Diamond\phi$$

I'll add a final principle for $\Diamond$,Äîa version of G,Äîin a moment.

For $\uparrow$ and $\downarrow$, I'll also add the following obvious principles.

$$\uparrow \Box \downarrow \phi \leftrightarrow \phi$$

$$\uparrow \Box \forall x (\downarrow \Box \phi \rightarrow \phi)$$

$$\downarrow \neg \phi \leftrightarrow \neg \downarrow \phi$$

and:

$$\uparrow \phi \leftrightarrow \phi$$

for $\phi$ without occurrences of $\downarrow$

Since $\Diamond$ only concerns the sense in which the interpretation of $\eta$ and $P$ can be extended, we will want to keep the interpretation of $R$ and $N$ fixed across interpretational possibilities.

So, we add:

(Strong Stability$_R$)

$$R(x,y) \rightarrow \Box R(x,y)$$
$$\neg R(x,y) \rightarrow \Box \neg R(x,y)$$

and:

(Strong Stability$_N$)

$$N(x) \rightarrow \Box N(x)$$
$$\neg N(x) \rightarrow \Box \neg N(x)$$

Since $\Diamond$ concerns the sense in which $\eta$ and $P$ are *expanded*, we want to make sure that as we move from one interpretational possibility to another, what counts as a property is preserved and what things a given property applies to is preserved.

So, we add:

(Stability$_P$) $$\forall x(P(x) \rightarrow \Box P(x))$$

and:

(Stability$_\eta$) $$\forall p(x\eta p \rightarrow \Box(x\eta p))$$
$$\forall p(\neg(x\eta p) \rightarrow \Box\neg(x\eta p))$$

where $p, q, r, ...$ etc range over properties.

What about the domains of our possible interpretations?

I want to be as flexible as possible here, and so the only constraint I'm going to impose is that as we move from one possible interpretation to the next, we don't lose objects.

(NEC-E) $$\forall x \Box Ex$$

(It turns out that it is consistent with the theory I'm articulating both that the domain of all possible interpretations is the same and that it grows.)

# Property Collapse

Next, we have Property Collapse$^\Diamond$ which says that any definition could define a property.

(Property Collapse$^\Diamond$) $$\uparrow \Diamond \exists p \Box \forall y \Box (y \eta p \leftrightarrow \downarrow \phi)$$

To explain the final principle, I need to introduce the language of typed properties over $\mathcal{L}$.

We add to $\mathcal{L}$ a stock of second-order monadic variables $X, Y, Z, \ldots$ etc, which take the position of ordinary monadic predicates. So, $Xx$ is well-formed and read "$x$ is $X$". I will also take $X = Y$ to be well-formed and read "to be $X$ is to be $Y$".

Let $\mathcal{L}^2$ be this language.

The main principle of typed property theory in $\mathcal{L}^2$ is the typed comprehension schema. It says, for any condition in $\mathcal{L}^2$ that there is a property that applies to all and only the $\phi$s.

(Comp$_{\mathcal{L}^2}$) $$\forall\vec{x}\forall\vec{X}\exists Y\forall y(Yy \leftrightarrow \phi)$$

where $\phi \in \mathcal{L}^2$ with free variables among $\vec{x}, y, \vec{X}$.

Now: just as the axioms of ZFC holding in the potential universe of sets is a gold standard for set potentialism, I take it that the instances of $\text{Comp}_{\mathcal{L}^2}$ holding in the universe of potential universe of properties is a gold standard for property potentialism.

We can make this precise by defining a translation from $\mathcal{L}^2$ to $\mathcal{L}_\eta^\Diamond$ that interprets second-order variables as ranging over potential properties.

- $(x = y)^\Diamond = x = y$, $R(x, y)^\Diamond = R(x, y)$, and $N(x)^\Diamond = N(x)$
- $(X = Y)^\Diamond = x_X = x_Y$
- $(Xy)^\Diamond = y \eta x_X$
- $^\Diamond$ commutes with the connectives
- $(\exists x \phi)^\Diamond = \Diamond \exists x \phi^\Diamond$
- $(\exists X \phi)^\Diamond = \Diamond \exists x_X (P(x_X) \wedge \phi^\Diamond)$

In line with set potentialism, I'll call $\phi^\Diamond$ the **modalisation** of $\phi$.

I now want to show that the theory I've outlined meets the gold standard. But first I need one more principle for $\Diamond$.

The G principle of modal logic is this:

$$\Diamond\Box\phi \rightarrow \Box\Diamond\phi$$

It basically says that modal space looks the same in all directions.

This turns out to be much too strong for my purposes.

Instead, I'm going to adopt its restriction to modalised formulas.

$$(\mathrm{G}^{\Diamond}) \qquad \forall \vec{x}_{\vec{X}}(\Diamond\Box\phi^{\Diamond} \rightarrow \Box\Diamond\phi^{\Diamond})$$

where $\phi \in L^2$ second-order free variables are among $\vec{X}$.

I won't explain why, but this very roughly just tells us that properties with the same application conditions are available in all directions. That you can't find a possible interpretation which rules out there being a property with some given application conditions.

Let T be the theory with all the previously mentioned principles apart from Property Collapse$^{\Diamond}$.

Some results...

First, we can show that the modalisations of claims in $\mathcal{L}^2$ behave logically just like those claims.

Where $\phi$ is a sentence of $\mathcal{L}^2$, we have:

**Theorem**

*$\phi$ is provable in second-order classical logic (without $\mathrm{Comp}_{\mathcal{L}^2}$) just in case $\phi^\Diamond$ is provable in $\mathrm{T}$.*

Theorems like this are usually called *mirroring theorems*. So I'll call this the mirroring theorem!

Second, we can show that with Property Collapse$^\Diamond$ we can meet the gold standard.

**Theorem**

*$\mathrm{T}$ + Property Collapse$^\Diamond$ proves $\phi^\Diamond$ whenever $\phi$ is an instance of $\mathrm{Comp}_{\mathcal{L}^2}$.*

Moreover, T + Property Collapse$^{\Diamond}$ integrates nicely with set potentialism. (At least, formally...)

Recall its central principle:

(Set Collapse$^{\Diamond}$) $\quad\quad \uparrow \Diamond \exists s \forall y (y \in s \leftrightarrow \phi \wedge \downarrow Ey)$

If we interpret $N$ as a predicate for being a set and $R$ as set membership and we add the appropriate set potentialist specific principles we can get a theory that interprets second-order ZFC!

So far so good! Success!

OK, let me now ruin all my good work.

Some of us would like use properties to make sense of classes in set theory. If we're potentialists about properties, we're presumably potentialists about sets too. So, we'd like to make sense of properties of sets.

The results I just mentioned show that this is possible! The problem is that they rely on positing a deep asymmetry between potential properties and potential sets. And I just don't think that asymmetry is justified. Let me explain.

So, let's assume that $\mathcal{L}$ is $\mathcal{L}_{\in}$ and therefore that we're working with *Set* and $\in$.

Then an assumption of T is that *Set* and $\in$ are strongly stable: that they have the same interpretation in all possible interpretations.

A weaker set of principles that would have sufficed for the theorems (together with a slightly different modalisation translation) is what I'll call **coherence**.

(Coherence$_{\in, Set}$)
$$\Diamond x \in y \to \Box \Diamond x \in y$$
$$\Diamond Set(x) \to \Box \Diamond Set(x)$$

And it turns out that without $\text{Coherence}_{\in,Set}$ or strong stability, things drastically fall apart.

Here's one way to see that. Consider the property of being a set. It has two obvious definitions.

$$\Box\forall x(x\eta p \leftrightarrow\downarrow Set(x))$$

$$\Box\forall x(x\eta p \leftrightarrow\downarrow \Diamond Set(x))$$

If *Set* only applies to the sets in the domain of a given possible interpretation, for example, then the first definition will not result in the property of being a set.

Similarly, if every object not in the given possible interpretation is a set in some extended interpretation, then the second definition would result in an almost universal property!

If we want even very simple properties of sets—like the property of being a set!—then we need Coherence$_{\in, Set}$.

The problem is that T + Property Collapse$^\Diamond$ requires that properties not obey coherence. That is, T + Property Collapse$^\Diamond$ is inconsistent with:

(Coherence$_{\eta,P}$)
$$\Diamond x\eta y \to \Box\Diamond x\eta y$$
$$\Diamond P(x) \to \Box\Diamond P(x)$$

The consistency and strength of Property Collapse$^\Diamond$, in particular, depends on failures of Coherence$_{\eta,P}$.

So, we need coherence for potential sets. But we can't have coherence for potential properties. What should we do?

Some set potentialists talk as though the potential sets are coherent. If that's right, then they face the challenge of explaining why, in spite of this, the potential properties are not.

I don't see how they might do this.

My own view is that, on the interpretational reading of potentiality, there simply isn't a good sense in which the potential sets are coherent.

On the interpretational understanding, I think the only plausible view is that there is nothing more to being a set than being a set *in some interpretation*. Sets have no intrinsic nature, and absolutely anything can be a set.

For my money, set potentialism on the interpretational understanding is best seen as a version of eliminative structuralism.

Let me end with an informal conjecture.

The preceding observations lead to thought that the amount of information encoded in a given interpretational possibility is relatively speaking small.

No interpretational possibility encodes membership for all the possible sets, nor application for all the possible properties.

So:

**Conjecture:** because interpretational possibilities encode a relatively small amount of information, property potentialism (of the kind I'm interested in) is bound to be **weak**. In particular, it won't be able to deliver enough potential properties to satisfy even all $\Pi^1_1$ instances of the typed comprehension schema (over $\mathcal{L}$).

However:

**Claim:** we can, with the right set up, get enough potential properties to satisfy the predicative instances of the typed comprehension schema (over $\mathcal{L}$). Indeed, we can get more than that, but I'm not sure I'm overly excited about that bit more.

# End!

Thanks!