Work in progress on the iterative conceptions of set

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1 The Weak and Strong Iterative Conceptions of Set

The Weak Iterative Conception holds that we (i) start with a set of objects, and (ii) using some set-theoretic operations, form new sets from old. There are no other sets.

The Strong Iterative Conception holds that we (i) start with some set of objects, and (ii) successively form *all possible sets* at each additional stage. There are no other sets.

Iterative conceptions are designed to:

1. Motivate a "nice" theory of sets.
2. Tell us why paradoxical collections do not form sets.

2 Some theories

Definition 1. *We will consider the following theories and axioms in* $\mathscr{L}_\in$*:*

(i) ZFC *(with the axiom of choice rendered as the claim that every set can be well-ordered).*

(ii) ZFC− *is* ZFC *with the powerset axiom deleted.*

(iii) ZFC^- *is* ZFC− *with the axiom schemes of Collection and Separation added.*

(iv) Count*; the axiom that all sets are countable.*

(v) Projective Determinacy (or PD*) is the schema of assertions stating that every projectively definable class of reals has a winning strategy.*

Theorem 2. *(Folklore[1]) Second-order arithmetic and* $\mathsf{ZFC}^- + \mathsf{Count}$ *are bi-interpretable.*

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[1] Although the theorem is folklore, it is very nicely presented in §5.1 of Regula Krapf's PhD thesis [Krapf, 2017].

3 Lin

Definition 3. *Extensional plural logic has the axioms (again, we give axioms informally, suppressing the formal details, see [Linnebo, 2014]):*

(i) A principle of extensionality for plurals (that if two pluralities xx and yy comprise the same things, then anything that holds of the xx also holds of the yy and vice versa).

(ii) An impredicative comprehension scheme:

$$\exists xx \forall y \big(y \prec xx \leftrightarrow \phi(y)\big)$$

for any ϕ in $\mathscr{L}_{\in,\prec}$ not containing xx free.

Definition 4. *[Linnebo, 2013] (here we follow [Scambler, 2021]'s presentation)* Lin *is the following theory in $\mathscr{L}^{\Diamond}_{\in,\prec}$:*

(i) Classical first-order predicate logic.

(ii) Extensional plural logic.

(iii) Classical S4.2 *with the Converse Barcan Formula added.*

(iv) The Axiom of Foundation (rendered as normal using solely resources from $\mathscr{L}_{\in}$).

(v) Extensionality (again using solely resources from $\mathscr{L}_{\in}$).

*(vi) (*Collapse$^{\Diamond}$*) The principle that any things (at a stage) could form a set:*

$$\Box \forall xx \Diamond \exists y \Box \forall x (z \in y \leftrightarrow z \prec xx)$$

(vii) Stability axioms for $\prec$ and $\in$ (these mirror the necessity of identity/distinctness):

- $x \in y \rightarrow \Box(x \in y)$
- $x \notin y \rightarrow \Box(x \notin y)$
- $x \prec yy \rightarrow \Box(x \prec yy)$
- $x \not\prec yy \rightarrow \Box(x \not\prec yy)$

(viii) Two principles of plural definiteness:

- ***Weak Plural Definiteness***: $(\forall x \prec yy)\Box\phi(x) \rightarrow \Box(\forall x \prec yy)\phi(x)$
- ***Strong Plural Definiteness***: $(\forall xx \prec yy)\Box\phi(xx) \rightarrow \Box(\forall x \prec yy)\phi(xx)$ *(where $xx \prec yy$ holds just in case the xx are a subplurality of the yy, i.e. every xx is a yy)*

(ix) The axiom that there could be some things comprising all and only the natural numbers.

(x) The axiom that there could be some things that are all and only the subsets of a given set.

(xi) Every potentialist translation of the Replacement Scheme of ZFC.

(xii) A plural version of the Axiom of Choice "For any pairwise-disjoint non-empty sets xx, there are some things yy that comprise exactly one element from each member of the xx".

Definition 5. *[Linnebo, 2013] The potentialist translation of a formula in $\mathscr{L}_{\in}$ into a language containing $\langle m \rangle$ is obtained by substituting every occurance of $\exists x$ by $\langle m \rangle \exists x \phi$ and every occurrence of $\forall x$ by $[m] \forall x \phi$.*

Theorem 6. *[Linnebo, 2010], [Linnebo, 2013]* ZFC *proves ϕ iff* Lin *proves $\phi^{\Diamond}$.*

Theorem 7. *[Linnebo, 2013] Within a model M of* ZFC*, the V_α under $\subseteq$ provide a model for* Lin*. Specifically a (M-proper-class-sized) Kripke frame validating* S4.3*.*

Theorem 8. *(*ZF*) For every set x there is an ordinal α such that $x \in V_\alpha$.*

The following things are highlighted by a "nice" iterative conception of set:

1. We are able to use Lin to explain why we do not get paradoxical collections, and which conditions do and do not form sets.
2. We are able to motivate a "nice" non-modal theory of sets T (in the sense that we can prove the potentialist translations of every sentence of T from within the modal theory). In this case, ZFC from Lin.
3. We are able, within T, to produce a suitably "nice" representation of the stages (in this case, the V_α).
4. We have a theorem asserting that every set is a member of some stage (from (2.)), in this case the theorem that every set belongs to some V_α.

4 Sca

Definition 9. Sca *consists of the following axioms in $\mathscr{L}_{\in,\prec}^{\Diamond,\langle h \rangle,\langle v \rangle}$:*

(i) Classical first-order logic.

(ii) Extensional plural logic.

(iii) Classical S4.2 *with the Converse Barcan Formula for every modality.*

(iv) ***Weak Plural Definiteness***

(v) The necessity of distinctness and stability axioms for $\prec$ and $\in$ (Scambler calls these 'definiteness axioms', but we'll follow [Linnebo, 2013]'s terminology).

(vi) The Axiom of Foundation (the standard one from ZFC*).*

(vii) Extensionality for sets (again, no different from ZFC*).*

(viii) ***Weakening Schemas****: $\langle h \rangle \phi \to \Diamond \phi$ and $\langle v \rangle \phi \to \Diamond \phi$, for every ϕ.*

(ix) ***Vertical collapse****: $\langle v \rangle \exists y \Box \forall z (z \in y \leftrightarrow z \prec xx)$.*

(x) The axiom that there could vertically be some things that necessarily comprise all and only the natural numbers: $\langle v \rangle \exists xx \Box \forall y (y \prec xx \leftrightarrow$ 'y is a natural number'$)$.

(xi) ***Subset Comprehension.*** *The axiom that its vertically possible to have some things that are vertically necessarily all the subsets of a set:* $\forall z \langle v \rangle \exists xx [v] \forall y (y \prec xx \leftrightarrow y \subseteq z)$.

(xii) ***Possible Generics.*** *The axiom 'If* $\mathbb{P}$ *is a forcing partial order and* dd *is some dense sets of* $\mathbb{P}$*, then it's horizontally possible that there is a filter meeting each dense set that is one of the* dd*'.*

(xiii) The plural version of the Axiom of Choice

Theorem 10. Sca *interprets* ZFC^- + Count *under the potentialist translation using* $\Box$.

Fact 11. *[Scambler, 2021]* Sca *interprets* ZFC *using the potentialist translation with the* $\langle v \rangle$ *modality.*

Fact 12. *[Scambler, MS] We can prove the potentialist translations of* ZFC^L *for every axiom of* ZFC *(our first-order rendering of the statement "*$L \models$ ZFC*").*

1. We are able to use Sca to explain why we do not get the paradoxical collections. Not only can the Russell set be formed over a stage, but we can always add forcing generics too. This has the consequence that several conditions often taken to form sets (e.g. "x is hereditarily countable") do not form sets.
2. We are able to motivate a nice theory using Sca, namely ZFC^- with inner models for ZFC (rendered as a schema about definable inner models).
3. However, though there is a consistency proof, there is no obvious "nice" representation of the stages.
4. We have no theorem from our "nice theory" (namely ZFC^- + Count) showing that every set is a member of some stage under Sca.

5 SteMMe (and variants)

Definition 13. *Steel's Multiverse Axioms are as follows:*

(i) The axiom scheme stating that if W *is a world, and* ϕ *is an axiom of* ZFC*, then* ϕ *holds at* W.

(ii) Every world is a transitive proper class.

(iii) If W *is a world and* $\mathbb{P}$ *is a forcing partial order in* W*, then there is a universe* W' *containing a generic for* W.

(iv) If U *is a world, and* U *can be obtained by forcing over some world* W*, then* W *is also a world.*

(v) If U *and* W *are worlds then there are* G *and* H *that are generic over them such that* $U[G] = W[H]$.

Definition 14. SteMMe *(for* ***Ste****el-****M****addy-****Me****adows) comprises the following axioms in* $\mathscr{L}^{\Diamond}_{\prec,\in}$

(i) Classical first-order logic.

(ii) Extensional plural logic.

(iii) ***The Ordinal Definiteness Schema:*** *This is the schema of assertions of the form* $\forall x$ *"*x *is an ordinal"* $\rightarrow \big(\Box\phi(x) \rightarrow \Box\forall y Ord(y) \rightarrow \phi(y)\big)$

(iv) ***Weak Plural Definiteness***

(v) Classical S4.2 *with the Converse Barcan Formula for every modality.*

(vi) The necessity of distinctness and stability axioms for $\in$ *and* $\prec$*.*

(vii) First-order ZFC*.*

(viii) ***Possible Set-Generics.*** *The axiom 'If* $\mathbb{P}$ *is a forcing partial order and* $\mathcal{D}$ *is a set of dense sets of* $\mathbb{P}$*, then it's possible that there is a filter meeting each dense set that is a member of* $\mathcal{D}$*'.*

(ix) The potentialist translations of every instance of the the Collection and Replacement schemas.

Lemma 15. SteMMe *implies that the predicate "is an ordinal" cannot change extension.*

Corollary 16. SteMMe *implies that* Collapse$^\Diamond$ *fails.*

Fact 17. SteMMe *implies that* ***Strong Plural Definiteness*** *fails.*

Conjecture 18. SteMMe *interprets* ZFC$^-$ + Count *under the potentialist translation.*

Conjecture 19. *Let* SteMMe$^-$ *be the result of removing the potentialist translations of Replacement are still provable (i.e.* SteMMe$^-$ *still interprets* ZFC$-$ + Count*).*

Conjecture 20. *Let* W *be (the necessitation of) the claim that "There is a proper class of Woodin cardinals". Let* SteMMe$^+$ *be the result of adding* W *to* SteMMe*. Then* SteMMe *interprets* ZFC$^-$ + Count + PD *(schematically rendered).*

6 Where forward?

We work over a ctm M.

Fact 21. *Let* M *be a ctm of* ZFC*. Then there are many non-interdefinable Cohen-generic reals we could add over* M*.*

Point. Accessibility need not be linear.

Fact 22. *Let* $M[G]$ *be obtained from* M *by the addition of a single Cohen real. Then there is a dense-order of forcing extensions (ordered by inclusion) between* M *and* $M[G]$*.*

Point. Accessiblity need not be well-founded.

Question. How to isolate the notion of an "iterative process"?

Definition 23. *A* potentialist system *is a collection of structures of the same type, ordered by a reflexive and transitive relation* $\subseteq$ *which refines the substructure relation.*

We are given some potentialist system S.

Definition 24. *A process of construction is a subframe* P *of* S *such that the accessibility relation* R *is well-founded.*

Definition 25. *We say that a process* P *is full iff for every set* x *in* S*, there is a world* W *in* P*,* $x \in W$*.*

Question. Can we come up with theories T that (i) imply that every set is countable, and (ii) will (like ZFC), get us the result that there is a full process of construction to which every set belongs?

References

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