Engineering Set-Theoretic Concepts

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Introduction

- This talk is about a mini-book I've been writing on conceptual engineering and set theory.
- One thing I want to do with the book is provide an intuitive account of some of the very technical philosophy of set theory that's happened in the last 20 years or so.
- I want to do two things in this talk:

> **Aim 1.** Give you a **commercial** for the book, and the idea that not only have conceptions of set changed in the past, but we are *now* facing our own **choice-points**.

> **Aim 2.** Outline some of the **mathematics** behind what I'm doing there, and some of the **open questions** for the future.

- **Note:** Let me know if you'd like to look at the draft of the book!

Contents

1 Concepts, conceptions, and conceptual engineering

- **Conceptual engineering** is the field of philosophy that concerns itself with conceptual change and related issues in the philosophy of language.
- Helpful here will be a distinction pointed to by [Incurvati, 2020] between *concepts* and *conceptions*.
- A *conception* is an account of what the sets are like that is used to motivate a *theory*.
- **Example.** You and I can have different conceptions of **fairness** (say whether I get a promotion) I think it should be determined by **outcome** and you think it should be determined by **effort**.

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- But you and I can **disagree** on a conception of a concept without thereby holding that one of us doesn't understand the term.
- I'll speak of **fundamental principles**, principles that are taken to be important to a concept/conception.
- Let's look at some **conceptions of SET**.

2 Conceptual engineering has happened: The iterative and logical conceptions of set

- **What is a set**? (i.e. What are the **fundamental principles** for SET?)

  Definition 1. (Informal) A *set* is a kind of *collection* that is:

  - **Extensional.** Sets with different members are non-identical, and sets with the same members are identical.
  - **Objectual.** Sets are *objects* over and above their elements.

- Let's note first that we have already had some some concept/conception **shift** by adopting SET **at all**.
- We have the «set-theoretic conception» of COLLECTION.
- Collections can be both **non-objectual** (e.g. pluralities) or **intensional**.
- So moving to SET is already a **substantial** piece of engineering!
- I want to consider how SET gets engineered, much has already been done (e.g. [Incurvati, 2020]).
- We started with:

> The *«naive conception»* of SET holds that sets are extensions of **arbitrary** predicates.

- As part of the «naive conception» we have the idea that the **naive comprehension schema** is true: $\exists x \forall y \big(y \in x \leftrightarrow \phi(x)\big)$
- But as we know this leads to **contradiction** via Russell's paradox and the condition $\phi(x) =_{df} x \notin x$.
- A diagnosis from [Incurvati, 2020], Russell's Paradox results from the way that the Naive Comprehension Schema allows for the following two fundamental principles for SET:
  - **Universality.** A concept C is universal iff there exists a set of all the things falling under C. ([Incurvati, 2020], p. 27)
  - **Indefinite extensibility.** A concept C is indefinitely extensible iff whenever we succeed in defining a set u of objects falling under C, there is an operation which, given u, produces an object falling under C but not belonging to u. ([Incurvati, 2020], p. 27)

  Two conceptions of set that have arisen in response:

> The *«iterative conception»* of SET holds that sets are formed in stages, starting from some **given** sets and then **collecting together** sets **available** at previous stages.

> The *«logical conception»* of SET holds that sets are extensions of **well-defined** predicates.

- The «iterative conception» gives up **universality** and the «logical conception» gives up **indefinite extensibility**
- (**Note:** I'm assuming that under the «logical conception» $x = x$ is always well-defined, there's lots of sharpenings e.g. «stratified conception», «iterative property conception» that get us this.)

- Compare with [Scharp, 2013]'s «ascending conception» and «descending conception» of TRUTH.
- [Incurvati, 2020] suggests that we pursue a strategy of **inference to the best conception**—compare the various conceptions of SET and their theoretical virtues.
- Part of these could involve e.g.
  - Explanation of the paradoxes.
  - Motivation of a nice theory of sets.
  - Respecting foundational constraints (e.g. provide a **Generous Arena**, give a good **Theory of Infinity**—cf. [Maddy, 2017] and [Maddy, 2019], I go over these in the book).
- The «iterative conception» splits further...

> The *«strong iterative conception»* of SET holds that sets are obtained in a sequence of stages. At each additional stage we form **all possible subsets** of sets available at previous stages.

> The *«weak iterative conception»* of SET also holds that sets are formed in stages. Sets are formed by **collecting together** sets at previous stages. However **we leave it open** whether or not we get **every possible subset** of what we have at a stage immediately after the current one.

- Each you can think of **modally**, the **stages** are **worlds** and the **collecting operation** gives **accessibility**.
- The «strong iterative conception» is familiar: Define V_α in the usual way.
- The «weak iterative conception» is less familiar, but occurs a lot.
- e.g. 1. The hereditarily α-sized sets H_α (generalised to ordinals).
- e.g. 2. The constructible universe and the L_α.
- **Note:** Sometimes you can **recover** the «strong iterative conception» e.g. If $V \models \mathsf{ZFC}$ then $L \models \mathsf{ZFC}$. But not **always** e.g. contrast $V_{\omega+\omega}$ with $L_{\omega+\omega}$.
- **Note:** We can make this **more** fine-grained, it need not be **linearly ordered** (treat each **formula** $\phi(x)$ as **its own operation**).

3 The absoluteness conception of maximal iterative set

- Unfortunately, the «iterative conception» is (probably) **consistent** but **defective**.
- We set theory to provide a **Theory of Infinity**:
  - Do **large cardinals exist**?
  - What is the behaviour of the **continuum function**?
- The «iterative conception» tells us **almost nothing** here.
- One thing that has happened is that many set theorists have moved to the «maximalist conception» of ITERATIVE SET.

> The *«maximalist conception»* adds the fundamental principle that there should be **as many sets as possible**.

- **Problem:** There are **all sorts** of maximality principles, and many disagree with each other (see [Incurvati, 2017] for a survey).
- For a simple example, CH can be seen as maximising (lots of **sets of reals**!) and so can $\neg\mathsf{CH}$ (lots of different kinds of **function**!).

- So we need to **sharpen** further.
- There's **lots** of ways we could go here. Here's one::

> The *«absoluteness conception»* holds that if there **could** be a set such that ϕ then there **is** a set such that ϕ.

- OK what does it mean for a set to be **possible** here?
- I'll take this to mean: Could be obtained either by **viewing V as a set** (I'll call this **climbing**) or by **moving to a forcing extension**.
- **Note:** Don't **freak out**, this can all be coded up! (cf. [Antos et al., 2021]).

> **Forcing Absoluteness.** If there is a **forcing** extension with a set such that ϕ, then there is a set such that ϕ.

> **Climbing Absoluteness.** If there is a **climbing** extension with a set such that ϕ, then there is a set such that ϕ.

- Let's **restrict** to Σ_1-sentences, since we can clearly run into issues with Σ_2-sentences (e.g. both CH and $\neg\mathsf{CH}$ are Σ_2).
- This looks **promising**!
- Presumably it's possible for there to be **uncountable cardinals** and **inaccessible cardinals**, by making Ord into a set.
- So we get **large cardinals** (given suitable possibility axioms).
- We also get resolutions to CH (in the negative) via **bounded forcing axioms** that have absoluteness characterisations.
- e.g. BPFA can be stated as the claim that if ϕ is a Σ_1 sentence with parameters from $\mathcal{P}(\omega_1)$, then if ϕ holds in a forcing extension obtained by proper forcing, then ϕ holds.
- So we seem to be making **some** progress.

4 A 'new' kind of paradox

- Unfortunately the «absoluteness conception» of MAXIMAL ITERATIVE SET is **inconsistent**.
- **The Cohen-Scott Paradox** begins by observing that by **Climbing Absoluteness**, there should be lots of **uncountable sets** and **large cardinals**.
- But also, by **Forcing Absoluteness** any particular set x you consider should be **countable**.
- Take any uncountable set x.
- By forcing, there is a bijection $f : x \to \omega$ in a forcing extension.
- By **Forcing Absoluteness** there is such a bijection $f : x \to \omega$.
- Contradiction!
- **OK:** What has gone wrong here?
- On the one hand **Climbing Absoluteness** pushes us to say that there are lots of **large cardinals** (any uncountable cardinal is large for me).

- On the other hand **Forcing Absoluteness** just wants to **kill off** the idea that cardinals have closure properties.
- (**Note:** No-one really runs into this paradox in formal work quite like we did with **Russell**. Set theorists are not dummies, and they see this problem a mile off...)

5 Contemporary engineering

- Like with the «naive conception» of SET and TRUTH we have two principles that come into conflict (**Forcing Absoluteness** and **Climbing Absoluteness**).
- **Option A.** Adopt the «climbing absoluteness conception», the «strong iterative conception», and incorporate as much **Forcing Absoluteness** against this.
- This motivates ZFC plus large cardinals and (bounded) forcing axioms.
- **Option B.** Adopt the «forcing absoluteness conception», the «weak iterative conception», and incorporate **Climbing Absoluteness** against this.
- This motivates ZFC minus Powerset plus "Every set is countable" plus Weak Reflection (to transitive sets, rather than V_α), call this ZFC^-_{Ref}
- **Challenge.** How to get a strong theory for this conception?

  Definition 2. *Extreme Inner Model Hypotheses.* The *Extreme Inner Model Hypothesis for* T or $\mathsf{EIMH}^{\mathsf{T}}$ states that if a first-order sentence $\phi(\vec{a})$ in the parameters $\vec{a}$ in V is true in a definable inner model $I^* \models \mathsf{T}$ of an outer model $V^* \models \mathsf{T}$ of V obtained by a definable pretame class forcing, then $\phi(\vec{a})$ is already true in a definable inner model $I \models \mathsf{T}$ of V. We shall use EIMH^- and EIMH^-_{Ref} to denote the EIMH for ZFC^- and ZFC^-_{Ref} respectively.

  Theorem 3. *[Barton and Friedman, Ms] The* EIMH^-_{Ref} *is inconsistent.*

  Definition 4. *Ordinal Inner Model Hypotheses.* The *Ordinal Inner Model Hypothesis for* T or $\mathsf{OIMH}^{\mathsf{T}}$ states that if a first-order sentence $\phi(\vec{a})$ with **<u>ordinal</u>** parameters $\vec{a}$ in V is true in a definable inner model $I^* \models \mathsf{T}$ of an outer model $V^* \models \mathsf{T}$ of V obtained by a definable pretame class forcing, then $\phi(\vec{a})$ is already true in a definable inner model $I \models \mathsf{T}$ of V. We shall use OIMH^- and OIMH^-_{Ref} to denote the OIMH for ZFC^- and ZFC^-_{Ref} respectively.

  Theorem 5. *[Barton and Friedman, Ms] The* OIMH^-_{Ref} *is consistent relative to* ZFC + PD.

  Theorem 6. *[Barton and Friedman, Ms] The* OIMH^-_{Ref} *implies that* $0^\sharp$ *exists (and hence that* ZFC *plus large cardinals is true in many inner models).*
- But can this be given a reasonable **stage theory**? The V_α are not available!
- Normally, we see **multiversism** and **universism** as claims about **ontology**—there **is** (**not**) a set-theoretic universe that is thus and so.
- My contention: **We can see some of these views as providing a stage theory for the «forcing absoluteness conception»**.
- [Steel, 2014] Worlds are proper class models of ZFC, and accessibility is given by forcing.
- [Scambler, 2021] Worlds are models of ZFC (actually he uses something second-order), and accessibility is either by adding ranks or forcing.
- We can continue our strategy of pursuing **inference to the best conception**.
- Each resolves the defect with respect to **Theory of Infinity** to a greater/lesser extent.
- But there are other trade offs to be made (e.g. with respect to **Generous Arena**—foundations looks very different on each approach).

6 Conclusions and open questions

- I think there's an argument to be made that we are at a **conceptual crossroads**.
- Some questions:
  - **Question 1.** What about **within** ZFC? e.g. under the «climbing absoluteness conception» . This might also naturally be seen as conceptual engineering...
  - **Question 2.** How to handle the model theory/explain the «weak iterative conception» in more detail? Should it be **well-founded**?
  - **Question 3.** Is there a correlate for some theory of the «forcing absoluteness conception» and $\mathsf{ZFC} \vdash \forall x \exists \alpha (x \in V_\alpha)$?
  - **Question 4.** What does the foundations of mathematics look like under the «forcing absoluteness conception» ? (e.g. functional analysis etc.)

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