Is the radical multiverse view coherent?

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Introduction

On one view of mathematics, we can 'pick out' (more or less) determinate structures, e.g. the natural numbers or the universe of sets (perhaps up to height).

In set theory, perhaps the boldest challenge comes from the radical multiverse view (defended most notably by [Hamkins, 2012]). Roughly:

> (1.) There is a multiverse consisting of different universes of sets (and if we say ZFC, we consider the multiverse of universes satisfying ZFC)
>
> (2.) Any statements and structures that are not absolute between the universes ZFC are indeterminate.

However ever since it was first formulated, there have been concerns ([Koellner, 2013, Barton, 2016]) about the view's coherence: in particular

> **Question.** Does radical multiversism somehow (problematically?) diagnose *itself* as indeterminate?

Our main question today: **Can the radical multiverse view be stated coherently?** We will:

- consider some of the main choice points in formulating the view;
- argue that certain extreme version of multiversism (including one reading of [Hamkins, 2012]) faces a tension: the 'mathematical naturalism' that motivates the view is incompatible with the level of indeterminacy proposed.
- discuss two ways of resolving this tension:
  - embracing determinacy (cost: undermining some motivation for multiversism)
  - embracing indeterminacy (cost: view is no longer naturalistic)

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The Multiverse View: Some Choice Points

In the paper, we identify **several** choice points, but the ones we'll use here are:

> **Naturalism.** There is a default presumption in favour of taking well-established and well-entrenched mathematical (set-theoretic?) practice at face value (e.g. not to systematically reinterpret set-theorists or take them to be speaking in a systematically misleading way). All else being equal, we should favour philosophical views that endorse this (and the costs of reinterpretation are higher the more 'central' the part of set-theoretic practice we are dealing with).

> This abundance of set-theoretic possibilities poses a serious difficulty for the universe view, for if one holds that there is a single absolute background concept of set, then one must explain or **explain away as imaginary** all of the alternative universes that **set-theorists seem to have constructed**. This seems a difficult task, for **we have a robust experience in those worlds, and they appear fully set-theoretic to us**. The multiverse view, in contrast, explains this experience **by embracing them as real**... [Hamkins, 2012, p. 418]

Note that everyone has **something** to reinterpret (e.g. we talk about THE CONTINUUM all the time, but the multiversist thinks they have a response to this).

The second principle we'll need occurs in response to the **categoricity arguments**.

> The difference is that when a mathematical issue is revealed to have a **set-theoretic dependence**, as in the independence results, then the multiverse response is a careful explanation that **the mathematical fact of the matter depends on which concept of set is used**, and this is almost always a very interesting situation, in which one may weigh the desirability of various set-theoretic hypotheses with their mathematical consequences. ([Hamkins, 2012], p. 419)

> **Bridging.** If a sentence is indeterminate in the multiverse, then this indeterminacy is reflected in the *fact of the matter*.

Consider now:

> $\mathbb{N}$**-determinacy.** Every sentence (in a reasonable formal language, e.g. PA) about natural numbers are determinate.

The following is a philosophical corollary of **Bridging**:

> $\mathbb{N}$**-indeterminacy. Not** $\mathbb{N}$-determinacy.

> There seems little reason why two different concepts of set need to agree even on the concept of the natural numbers. Although we conventionally describe the natural numbers as 1, 2, 3, . . ., and so on, why are we so confident that this ellipses is meaningful as an absolute characterization? [Hamkins, 2012, p. 427]

Stating the view and problems with metatheory

Let's suppose that a radical multiversist utters:

"Consider the multiverse of models of ZFC"

What does "ZFC" mean there?

ZFC is a recursively axiomatized theory; saying what a model of ZFC is therefore requires some concepts from set theory and syntax (/arithmetic).

Dilemma: Do we have a grasp of these concepts that is **prior** to/independent of the multiverse?

First answer: Yes!

Problem 1: This contradicts **Bridging** and **$\mathbb{N}$-indeterminacy**.

Problem 2: If we have a prior grasp of e.g. arithmetic, this yields philosophical constraints that arguably undermine the multiverse conception in the first place. For instance, if we have a notion of an **arithmetically sound** model, why not restrict the multiverse to include only these?

Second answer: No!

Problem 1: Different 'elements' of the multiverse disagree about certain claims expressible in the language of arithmetic. For instance, the formula canonically expressing 'x is an axiom of ZFC' will have a different extension in different models of ZFC (e.g. in models with non-standard ω).

Problem 2: Actually, things seem worse than simple indeterminacy: there are models of ZFC in which $\neg Con(ZFC)$ is true. So there are some models of ZFC according to which the multiverse is empty.

There seems to be a fundamental tension in attempting to describe the multiverse as a whole vs. the idea that we can 'live in' particular models, each of which is on a par when it comes to doing mathematics.

The Algebraic Response

Algebraic response: Grab the horns of the "No!" option.

(Argued for in [Barton, 2016], but some of these ideas are definitely present in [Hamkins, 2012].)

We're not really interested in the multiverse as a specification of **ontology**, but rather the multiverse idea is to examine how things **could** look 'algebraically' from inside some structure or other.

Consider group theory. Groups can have **a bunch of different properties** (e.g. abelian or non-abelian) and we can consider **what can be constructed** from a given group (e.g. products, wreath products etc.).

We now take this to an **extreme**: Give me some structure M (any structure!) that models some axioms or other.

M has a conception of what ZFC is, and a conception of what the multiverse looks like.

Response to Problem 1: "ZFC" just denotes whatever M thinks ZFC is (i.e. ZFC^M).

Response to Problem 2: It's coherent to think that the multiverse is empty, this is one way things **could be**, there are other ways things could be (comparison: groups can be abelian or non-abelian).

Problems with truth

In order to see the problem we'll need a distinction:

Internal multiverse of M. The *internal multiverse of M* is the collection of all sets in M that M thinks model ZFC^M.

External multiverse of M. The *external multiverse of M* is the collection of models of ZFC^M that contain M.

Full multiverse of M. The *full multiverse of M* is the external and internal multiverse combined.

The internal multiverse can **always** be analysed from within M.

But really we want the algebraic perspective to let us see the **full** multiverse (e.g. forcing extensions).

Problem: There's really no way of knowing whether M **itself** is in its own full multiverse.

Dilemma: Does M **satisfy** ZFC^M?

If we need **external resources** to get a **grasp** on this, then the view is **not fully algebraic**.

If algebraic then we're OK, and this holds just in case for any universe there's a **fact of the matter** whether M satisfies ZFC^M.

But that needn't be case:

Theorem 1. *[Hamkins and Yang, 2013, Theorem 12] Assume Con("Ord is Mahlo") [this more than suffices, but we make it for clarity of exposition]. Then there are M_1 and M_2 such that:*

1. *M_1 and M_2 model* ZFC *(computed in the ambient universe).*
2. $(V_\delta, \in)^{M_1} = (V_\delta, \in)^{M_2}$
3. *M_1 believes V_δ models* ZFC
4. *M_2 believes V_δ does not model* ZFC

What's going on is that M_1, M_2, and V_δ all **agree** on what ZFC is.

It's just that the **satisfaction classes** can be disagreed upon.

So whether a model like V_δ even satisfies some of its own sentences of ZFC is **external** to the model.

And it can be **indeterminate** whether some M, with its own conception ZFC^M of ZFC, indeed satisfies ZFC^M.

The **full multiverse** can be indeterminate, even if the **internal multiverse** is determinate.

The fixes (?)

The issues with satisfaction depended upon looking at satisfaction classes for **non-standard** axioms.

> **Idea:** Look at the **determinately standard** axioms of ZFC—those that can be **fixed as standard externally**.

How exactly we should analyse this idea is a **difficult question**, but grant some **standardness** distinction for now.

The **determinately standard** axioms of ZFC **are** agreed on throughout the multiverse.

So fixing on these **avoids** the problem with satisfaction.

But we **know** that (given **Bridging**) there's **no** determinate conception of **standard** for the radical multiversist, so how to latch on?

$\mathbb{N}$-determinacy

Option 1: *Just accept the natural numbers are determinate*!

(FWIW: This is our **preferred** solution.)

We give up on $\mathbb{N}$-indeterminacy, and just accept that there's a determinate conception of **standard axiom of** ZFC.

> **The Standard Multiverse.** The multiverse is the collection of all structures that satisfy the **standard** axioms of ZFC (hang the non-standard ones!).

Of course, the question then is:

Question. Why **care** about the (algebraic) multiverse conception? It gets **arithmetic wrong**!

Response. Sure, it gets arithmetic wrong. But its still a **useful** conception **worthy** of study.

Studying the **independence of consistency**, **non-standard analysis**, **non-well-founded ultrapowers**! It's a **rich** conception.

But the natural numbers **are** determinate, and thus we **give up** on **Bridging** and $\mathbb{N}$**-indeterminacy**.

Non-naturalism

Option 2: *Give up on naturalism!*

OK, let's say that you want instead to **keep** $\mathbb{N}$-indeterminacy.

We still want some way of latching on to **determinately standard**.

We suggest the following idea:

Consider the axioms of ZFC that I could **feasibly write down**.

(**Note:** By "feasible" we do **not** mean polynomial-time writeable, we mean some (possibly inscrutable) sense that the physical world fixes.)

The Feasible Multiverse. The multiverse is then the collection of all models of **every feasible fragment** of ZFC.

Now we keep $\mathbb{N}$**-indeterminacy**...

...but **Naturalism** seems to be adversely affected.

Consider, for example:

"The Δ_n-satisfaction predicate is Δ_{n+1}-definable!"

Strictly speaking, this is going to fail for some n (when feasibility runs out).

So some kind of **non-face value** reinterpretation is required.

Note the parallels with **ultrafinitism**!

Conclusion

We think that the Radical Multiverse view needs some **relaxing**.

But there're **options** here, and in particular the role of the radical multiverse conception of set seems especially interesting **when we allow** $\mathbb{N}$**-determinacy.**

Maybe even this makes the view more **palatable** and **interesting** for mathematics **more widely**?

References

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