Week 12. The Quest For New Axioms

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Recap and this week

Last week we looked at *structuralism*, the idea that the subject matter of mathematics concerns *structures* rather than *objects*.

This week we'll discuss how one particular important structure—the cumulative hierarchy—has some interesting questions surrounding it.

I want to do three key things:

1. Provide some of the mathematics behind independence in set theory.
2. Introduce you to some key positions in the ontology of set theory.
3. Introduce you to some of the debates surrounding the *justification* of set-theoretic axioms.

1 Review: The Iterative Conception

Let's first start with the contemporary view infinity.

The idea is the following *Iterative* Conception of Set.

The thought is that sets a formed in *stages*.

I start at stage 0 with nothing.

I then (at stage 1) take all the sets I can form out of things at stage 0, and collect them together (this is just $\emptyset = \{\}$).

I then (at stage 2) take all the sets I can form out of things at stage 1, and collect them together (this is $\{\emptyset, \{\emptyset\}\}$).

I then (at stage 3) take all the sets I can form out of things at stage 2, and collect them together (this is $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$).

...and so on....

I then assume I have an infinite stage, called stage ω. It's just everything I formed at some stage n.

I take all the subsets I can form out of stage ω, to get to stage $\omega + 1$, this is just the set of all subsets (powerset) of stage ω.

At stage $\omega + 2$, I take the powerset of stage $\omega + 1$.... etc.

I then assume I can collect in the limit, to get a stage $\omega.2$.

I take the powerset to get to stage $\omega.2 + 1$...

... and so on ...

This *rough* idea is what lies behind the *iterative conception of set*.

It's *iterative*, because you can think of it as iterating the operations of powerset and union to infinity (and beyond)!

We saw how this was interlinked with the axioms of ZFC

We can formalise the idea in ZFC (**Z**ermelo-**F**raenkel Set Theory with the Axiom of **C**hoice).

From within ZFC (in fact ZF) we can define the following:

Definition. The *cumulative hierarchy of sets* is defined as follows:

$V_0 = \emptyset$

$V_{\alpha+1} = \mathcal{P}(V_\alpha)$ (at successor stages)

At limits λ:

$V_\lambda = \bigcup_{\beta<\lambda} V_\beta$

Fact. ZFC $\vdash (\forall x)(\exists \alpha) x \in V_\alpha$

So ZFC (in fact Z) formally codifies the idea that the sets are obtained by iterating the powerset operation and collecting at limits.

(**N.B.** In the above, I've used various symbols not in $\mathscr{L}_\in$. They can all be treated as defined symbols.)

2 The Continuum Hypothesis and Friends

We recall the following theorem:

Theorem. (Cantor's Theorem) For all x, $|\mathcal{P}(x)| > |x|$.

Observation. Assuming Z (i.e. not necessarily Replacement or Choice) there is an infinite hierarchy of cardinals.

In particular we know that ω is the smallest infinite cardinal, and that $\mathcal{P}(\omega)$ is larger.

But is there anything *in between*?

The Continuum Hypothesis. There is no cardinal number intermediate in size between ω and $\mathcal{P}(\omega)$. So $\omega = \aleph_0$ and $|\mathcal{P}(\omega)| = |2^{\aleph_0}| = |\mathbb{R}| = \aleph_1$.

Hilbert included proving (or refuting) the continuum hypothesis as number one on his list of problems presented to the International Congress of Mathematicians in 1900.

Fact. (Gödel, 1938) Assuming ZFC is consistent, you can build a model (called the *constructible* universe or L), such that $L \models$ ZFC + CH, and hence if ZFC is consistent, ZFC $\nvdash \neg$CH.

Fact. (Cohen, 1963) Assuming ZFC is consistent, you can build a model $L[G]$ (using *forcing*, by adding subsets of natural numbers) such that $L[G] \models$ ZFC + $\neg$CH, and hence if ZFC is consistent, ZFC $\nvdash$ CH.

Note: CH is *not* like consistency sentences. For example: If you're using ZFC, presumably you accept $Con(\mathsf{ZFC})$ (you probably shouldn't use a theory you have serious suspicions is ω-inconsistent if you're trying to represent mathematics). CH is a different matter. I can use ZFC, and not have any idea about the truth value of CH.

A different kind of independence: *large cardinals*.

These postulate the existence of sets that have nice 'largeness' properties, and do yield increases in consistency strength.

e.g.

Definition. A cardinal κ is *strongly inaccessible* iff κ is uncountable, regular (there's no function from a smaller ordinal unbounded in it) and a strong limit (if $\alpha < \kappa$, then $|\mathcal{P}(\alpha)| < \kappa$).

Fact. ZFC + "There is an inaccessible cardinal" implies $Con(\mathsf{ZFC})$ (and much more!).

Definition. A cardinal κ is *Mahlo* iff κ is inaccessible and there is a stationary set (i.e. a set that intersects every closed and unbounded subset of κ) of inaccessible cardinals below κ.

Fact. ZFC + "There exists a Mahlo cardinal" implies $Con($ZFC + "There is a proper class of inaccessible cardinals"$)$.

Fact. Assuming the relevant theories are consistent, the existence of a stronger large cardinal is independent from a weaker theory. (Why? *Hint:* Gödel.)

Two responses to the problem of independent sentences.

Universism. (Gödel) The iterative conception of set determines a perfectly precise universe. So independent questions have determinate answers, we just haven't found the axioms yet.

Multiversism. The independence results show that our thought is *indeterminate* and what we really investigate are just *models*.

For example, maybe CH does not have a determinate truth-value, and the problem is essentially solved.

Note: Multiversism is really a *family* of views (Are the subsets determinate? Are the ordinals determinate? Are the natural numbers determinate?)

There's lots more to be said, and a *huge* library of sentences independent from ZFC.

3 Justification of axioms

Key Question. Can we justify new axioms for set theory? If so, how?

Pen Maddy's Response. Let's distinguish between *two* kinds of justification:

Intrinsic justification is when a principle is supported by an intuitive underlying conception.

Extrinsic justification is when a principle is supported by the fact that it gives useful/successful consequences.

Perhaps there are *extrinsic* arguments for axioms even if there aren't *intrinsic* ones.

Question. Is it clear that intrinsic and extrinsic justification can be neatly separated this way?

Question. Are these really precise enough notions to do justificatory work?

Note: Gödel is very clear to talk about *verifiable* consequences when discussing extrinsic justification. Other authors opt for something slightly different (e.g. Maddy in *Defending the*

Axioms talks about 'deep' mathematics).

4 Questions and Discussion

Cluster: Justification.

Question. (Brian/Nicola) Can we separate intrinsic and extrinsic justification?

Question. (Julius/Nicola/Pietro) Does CH have interesting mathematical consequences? What if it didn't?

Question. (Ingvild) Does set theory really matter for mathematics?

Question. Is one of intrinsic/extrinsic justification more important?

Question. Is there such a thing as mathematical data? If so, how might we acquire it?

Question. What if some set-theoretic axiom had some *physical* consequences? Would that affect its prospects for justification?

Question / Optional Exercise. (Very tricky) Can you think of a *physical* consequence of the truth of the axioms of ZFC?

Hint. Think about the relationship between *truth*, *consistency*, and *finite sequences*.

Cluster: Ontology

Question. (Birgit/Julius/Brian) What does it mean to say that there's 'just one' universe of sets? Is there just one or many.

Question. (Pietro) Why is Gödel so concerned about distingushing the case of the 5th axiom of geometry from the situation with CH?

Question. (Nicola) Should we think of $\aleph_0$, $\aleph_1$ etc. as real numbers or 'constructs'?