Week 5. Intuitionism

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Recap

Last time we saw:

(1.) Hilbert's Programme attempted to handle issues of paradox by pressing a *finitism* and suggesting his programme to show consistency.

(2.) This was essentially sunk by the Incompleteness Theorems, but lived on in restricted forms.

This week, we'll look at the *Intuitionists*, and in particular Luitzen Egbertus Jan Brouwer (1881–1966), Arend Heyting (1898–1980), and Michael Dummett (1925–2011).

As we'll see, they had a very different attitude towards mathematics, in particular the *Law of Excluded Middle* ($\phi \vee \neg\phi$).

1 Intuitionism: The basic idea

Brouwer felt the formalist approach to mathematics did not respect its phenomenology.

Instead he proposed *Intuitionism*. We'll state it more precisely in a minute, but this is the idea that mathematics is somehow mind-dependent.

Kant writes the following in the *Proglomena*:

> Geometry bases itself on the pure intuition of space. Even arithmetic forms its concepts of numbers through successive addition of units in time.

So, for Kant, mathematics is *extracted* from sense experience. Brouwer takes up and develops this thought.

> The question where mathematical exactness does exist, is answered differently by the two sides; the intuitionist says: in the human intellect, the formalist says: on paper.

2 The epistemology and metaphysics of Intuitionism

Brouwer thought that Kant was wrong about geometry: There are non-Euclidean geometries, and so many different ways of representing the spatial structure of our experience.

He did think Kant was correct about arithmetic, however:

> This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts,..., the intuition of the bare two-oneness. This intuition of two-one-ness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers... (Brouwer, 'Intuitionism and formalism', 1913)

So, for Brouwer, our knowledge of arithmetic is obtained by reflecting upon the way that our experience is phenomenologically structured through time.

He also felt that we were able to get an idea of continuum from our phenomenology:

> Finally this basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i. e., of the "between," which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units. (Brouwer, 'Intuitionism and formalism', 1913)

So for Brouwer, our idea of the continuum flows from the idea that we can always interpose a point between any two others.

Remark. These are very Aristotelian ideas!

Brouwer held that the objects of mathematics are *mental constructions* derived from our *intuition*.

In this sense, we cannot say that a mathematical object exists until we have actually constructed such an object.

Such a construction will take the form of a proof (possibly non-formal!—bear in mind that mathematics is a process by which we intuit objects for Brouwer).

To bring everything together, we can define:

Brouwerian Intuitionism holds that:

(i) Mathematical reality comes into being with our acts of *mental construction*.

(ii) There is no distinction between what is true in mathematics and what has been proved.

(iii) There is no untensed truth.

3 Intuitionistic logic and mathematics

What kind of logic and mathematics does Brouwer's intuitionism support?

Brouwer came up with his own version of logic: *intuitionistic logic*.

One way of approaching this view is by interpreting the connectives as algorithmic claims about proofs (the so called Brouwer–Heyting–Kolmogorov interpretation).

A proof of $\phi \wedge \psi$ is a a pair consisting of a proof of ϕ and a proof of ψ.

A proof of $\phi \vee \psi$ is a proof of ϕ or a proof of ψ.

A proof of $\phi \rightarrow \psi$ is a construction that turns a proof of ϕ into a proof of ψ.

A proof of $\neg\phi$ is a construction that turns the assumption of ϕ into a contradiction.

A proof of $\forall x\phi(x)$ is a construction that produces $\phi(a)$ for any object a.

A proof of $\exists x\phi(x)$ is the specification of an object a and a proof of $\phi(a)$.

Note: Brouwer himself was *against* formalisation, and did not like this interpretation (he referred to it as "sterile").

Some features:

The Law of Excluded Middle Fails. The Law of Excluded Middle states that for any formula ϕ, $\phi \vee \neg\phi$ is a logical law. This fails in intuitionistic logic, since in order to use $\phi \vee \neg\phi$ I have to have *actually* proved one of ϕ or $\neg\phi$. But this isn't always the case in classical logic. Consider the following proof (given by Linnebo):

Theorem. (Classical logic) There are irrational numbers a and b such that a^b is rational.

Proof. Consider $\sqrt{2}^{\sqrt{2}}$. Is it rational? If yes, then $a = b = \sqrt{2}$ does the job. If no, then $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ do the job since:

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}.\sqrt{2})} = \sqrt{2}^{2} = 2$$

(by tedious algebraic manipulation)

But we haven't *actually* constructed such a pair of a and b here (I didn't show you which objects could be constructed that would do the job).

Double negation elimination fails. Double negation elimination is the law $\neg\neg\phi \rightarrow \phi$. But this fails in intuitionistic logic, since if I show you that from the assumption of $\neg\phi$ I can show a contradiction (i.e. $\neg\phi$ isn't provable) I haven't thereby exhibited a proof of ϕ.

Optional Exercise. Consider two common forms of proof by contradiction:

Reductio ad absurdum states that if I assume ϕ, and I prove a contradiction, I can infer $\neg\phi$.

The Principe of Indirect Proof (PIP) states that if I assume $\neg\phi$ and prove a contradiction, then I can conclude ϕ.

Question. Which one of those two does the Intuitionist reject? Why?

That's some features of the logic (underlying rules of inference) for the intuitionist.[1]

Arithmetic is largely the same, but with intuitionistic logic instead of classical and a conception of the numbers as forever in a process of construction.

Heyting Arithmetic has the same non-logical axioms as **PA**, but the underlying logic is intuitionistic.

This has implications for how we show generalised theorems. If I want to show $\forall x\phi(n)$, it's not enough to show that if I assume $\neg\forall x\phi(n)$ I get a contradiction, I need an actual method that will prove $\phi(n)$ for any particular n you give me.

Brouwer proposes a *complete rejection* of uncountable infinities. This is how he avoids the

[1]Careful with terminology! Some authors refer to this as reductio too. Precisely because intuitionists differ on the matter, I prefer to call them both proofs by contradiction but distinguish reductio and PIP.

paradoxes, talk of large infinite sets and paradoxical collections is completely meaningless.

There is then the question of how we deal with the *real numbers* (since these are often understood as *infinite* sequences).

Brouwer's response: We can think of these as *choice* sequences, they are not actually infinite, but rather ways of specifying continuations of finite sequences.

That's easy enough when we have a rule (what are called *lawlike* sequences), e.g. $0, 2, 4, 6, 8, ..., 2n, ...$

But what about seemingly *random* sequences e.g. $9, 2, 3, 5, 6, 8, ...$

These can be thought of as *free* choice sequences: Think of a never-ending sequences of coin-flips (or if you want base 10 notation, rolling a d10 die).

Upshot: Brouwerian Intutionism can be provided with its own interesting logic and mathematics.

4 Challenges for the view

Let's start with one strength of the view:

Benacerraf's Epistemological Challenge. Given that numbers are non-spatio-temporal, acausal entities, how do we gain knowledge of them?

Response. Numbers aren't non-spatio-temporal or acausal, we *literally* construct them, and they are just *part* of our experience.

Now let's move on to some more significant problems:

Provability Challenge. Surely there are unproved but provable propositions?

Response. What do you mean by "ϕ is provable"?

If you mean "there is a proof of ϕ" then there's no unproved but provable mathematical propositions.

If you mean "the possibility of proving ϕ has not yet been refuted" the fact that an unproved sentence is provable does not entail that there *exists* a proof awaiting discovery.

(Remember intuitionists *mean* something very different with their logic from the usual material interpretation! Things have to be *constructed*.)

Revisionary Challenge. Intuitionism is strongly revisionist about mathematical practice.

Depending on the kind of programme you want, this would have been utterly untroubling for Brouwer (in fact he quite liked this).

He was aiming to revise mathematics, and intuitionistic mathematics can be provided with corresponding formal theories (for e.g. number theory, analysis, and set theory).

Phenomenological Challenge. This talk of bare-two-one-ness is at best perplexing, and at worst incoherent (especially when we want a foundations for *mathematics*, which is meant to be especially secure).

Identity Challenge. Let's grant some kind of interpretation of bare-two-one-ness and mental construction. How many 2s are there? Is my 2 the same as your 2? What happens if we construct *the same* proof? How do we account for that claim?

Temporal Challenge. Wasn't it *always* the case (or not) that there were (or weren't) exactly two helium atoms orbiting the sun? What does any *actual* construction have to do with it?

(**Possible response.**) We're considering what *could* have been constructed. But this notion of ideal construction looks like it would need mathematical analysis, and this seems problematic from an intuitionistic perspective.

It thus seems that Brouwerian Intuitionism, whilst it has some strengths, is pretty problematic (philosophically speaking).

5 Dummett's Intuitionism: Meaning

Michael Dummett (1925–2011) proposed a different way of motivating intuitionism that (whilst still controversial) avoids some of this more problematic aspects of Brouwer's philosophical baggage.

Dummett's Intutionism arises out of considerations about *language* and *meaning*, rather than on the basis of an underlying *ontological account*.

Dummett is quite precise (once you've read things a few times), but it's a little difficult to untangle what he says (there's lots of moving parts), so let's walk through it slowly. Key is the claim that meaning is prior to questions of ontology:

> ...a philosophical account of thought can be attained through a philosophical account of language, and, secondly, that a comprehensive account can only be so attained. (Dummett, *Origins of Analytical Philosophy*, 1993)

Dummett's Meaning Principle. We can only come to an adequate account of (the) philosophy (of mathematics) through considerations of language and what we *mean* by our terms.

Indeed this is how he diagnosed some of the failings of Brouwer's account; he claims that Brouwer failed to take enough notice of linguistic considerations:

> traditional intuitionist accounts, which, notoriously, accord a minimum of importance to language or to symbolism as a means of transmitting thought, ... are constantly disposed to slide in the direction of solipsism. (Dummett, 'The Philosophical Basis of Intuitionistic Logic', 1975)

Two principles about language are key for Dummett's philosophy, one arising from the work of Frege, and the other Wittgenstein:

The first we have already encountered from the work of Frege:

Context Principle. The meaning of a word only has meaning in the context of a sentence.

So I can't just ask what the meaning of "3" is, I have to ask about the contexts in which "3" might legitimately appear.

We now ask:

Question. How can we analyse the meaning of words (in a particular context)?

Dummett followed the following (very Wittgensteinian) thought:

> The meaning of a mathematical statement determines and is exhaustively determined by its use. (Dummett, 'The Philosophical Basis of Intuitionistic Logic', 1975)

Let's call this the **Meaning as Use Principle**.

How should we understand this thesis? One option (we'll discuss next week):

Holism. The meaning of a term consists in its inter-theoretic connections with other terms, and permissions for use within them.

But he rejects this, to have this conception would require me to know the whole language, and that's clearly not how we operate.

Instead, for Dummett, to understand the meaning of a mathematical statement is just to *be able* to use the term correctly within communities of speakers in different contexts: How can we compute with numbers? How do we apply numbers to the real world and what kind of sentences do we assent to?

Given the Meaning as Use Principle, how should we understand truth? Dummett argues as follows:

> We are certain of the truth of a statement when we have conclusive grounds for it and are certain that the grounds which we have are valid grounds for it and are conclusive. (Dummett, 'The Philosophical Basis for Intuitionistic Logic', 1975)

In mathematics, what's the way of showing a fact conclusively? Why proof of course!

> What we actually learn to do, when we learn some part of the language of mathematics, is to recognise, for each statement, what counts as establishing that statement as true or as false. In the case of very simple statements, we learn some computation procedure which decides their truth or falsity: for more complex statements, we learn to recognise what is to be counted as a proof or a disproof of them...We must, therefore, replace the notion of truth, as the central notion of the theory of meaning for mathematical statements, by the notion of proof (Dummett, 'The Philosophical Basis for Intuitionistic Logic', 1975)

For Dummett then:

1. An understanding of the meaning of a statement consists in the capacity to recognise a proof of it.
2. An understanding of anything smaller than a sentence (e.g. the term "the least natural number") consists in the computational ways it can contribute to a proof of a sentence.

Note that this immediately suggests the Brouwer-Heyting-Kolmogorov (BHK) interpretation we discussed earlier!

Clarifications?

6 Dissolving some problems

The Phenomenological Challenge is gone: Whilst there are difficult questions about use, it is a lot clearer than the idea of bare two-one-ness, pulling difficult questions about the ontology of mathematics down in to more tractable ones about language use.

The Identity Challenge is also avoided: The question of whether two mental constructions are identical comes down to questions of possible use, there's just no ontological question to answer.

More generally Dummett has very little time for these purely ontological questions, referring for example to questions about realism as 'spurious'.[2]

For example even if the natural numbers exist "independently of the human mind" (and we can give meaning to that phrase!) we might still be intuitionists; the words we use mean the things they do insofar as we have capacities for use and proof.

So whether or not numbers are "out there" (or whatever) they are constructions insofar as our thought and language are concerned.

The Temporal Challenge is still open though.

This is a bit more difficult. Wasn't it *always* the case that there were (or weren't) two helium atoms circling the sun at some given distance 5 billion years ago?

> the appropriate generalisation of this [the account of truth via proof], for statements of an arbitrary kind, would be the replacement of the notion of truth, as the central notion of the theory of meaning, by that of verification; to know the meaning of a statement is, on such a view, to be capable of recognising whatever counts as verifying the statement. (Dummett, 'The Philosophical Basis for Intuitionistic Logic', 1975)

We might then say the following:

Possible Verification. The truth of a past-tense sentence consists of its being the case that someone suitably placed could have verified it.

With this in play, Dummett can say that there are two helium atoms because someone suitably placed could have verified (or falsified) that there are two such atoms.

Some interesting points here:

1. This allows Dummett to say that there's a large class of statements that the Platonist and Intuitionist can agree on; those that are *decidable* (could be verified by a proof in the relevant intuitionistic theory).

2. It perhaps opens him up to objections of the following form:

Isn't this 'ideal observer' highly mathematised?

Clearly they need to be seriously superhuman (check the number of atoms in a sphere five meters wide at the centre of the sun, for example).

But they can't be too superhuman...

A deity decides they want to create some numbers (say by observing intervals of space, or whatever). After half a second, they create 0. Then after another quarter of a second, they creates 1. Then after another eighth of a second they create 2,...and so on. After a full second, they've created infinitely many numbers (and can then perhaps 'observe' an undecidable sentence in that structure).

[2] e.g.

> Realism is a metaphysical doctrine; but it stands or falls with the viability of a corresponding semantic theory...the context principle repudiates semantics. That principle ... ought therefore not to be invoked as underpinning realism, but as dismissing the issue as spurious. (Dummett, *Frege: Philosophy of Mathematics*, 1991)

The point is, in order to explain what an idealised observer looks like, don't we need to pack in mathematical limits (e.g. can't observe infinitely many things).

But this ideal observer is meant to be grounding our possible patterns of use for mathematical subject matter, which in turn explains the meaning!

We seem to need the resources we're trying to explain in doing the explaining.

Let's be clear though: Whilst intuitionism faces cogent philosophical objections, the fruit it bore was impressive; it turns out (via something called the Curry-Howard correspondence) that intuitionistic logic is linked to kinds of type theory, and this has applications in functional programming.

7 Discussion

(Jens/Julius/Michel/Birgit/Haochong) What's so 'intuitive' about intuitionism?

(Fartein/Nicola/Pietro/Brian) Does invoking Husserl's 'transcendental subject' help Brouwer?

(Magnus) What is *truly mathematical language*?

(Haochong) Is logic part of mathematics or is mathematics part of logic?

(Nicola) Does Brouwer's view about truth-apt-ness carry over into language? (How about when we add Dummett's views in?)

(Jonas) Does Brouwer have a problem of communicability? (Does it get solved by Dummett's approach?)

(Jonas) Does Intuitionism solve problems of Hilbert's formalism?

(Jonas) Is it subject to the incompleteness theorems? (**Answer:** Yes, but is it so interesting for the intuitionist? e.g. $\phi \vee \neg\phi$ isn't valid so many intuitionistic theories are somewhat trivially incomplete.)

(Michel) What is the difference between Heyting's inuitionism and Brouwer's?