Week 9. Abstraction Reconsidered

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Recap

Last week we talked about mathematical *intuition*, and what epistemological fruits that might (or might not) deliver.

This week we'll examine a different approach based on reviving the logicist ideas of Frege and Russell.

1 Neo-Logicism: The Basic Idea

Frege's Logicism used the following principle:

$$\text{(Basic Law V) } \{x|F(x)\} = \{x|G(x)\} \leftrightarrow (\forall y)(F(y) \leftrightarrow G(y)))$$

This gave him the resources to show that the axioms of second-order Peano Arithmetic (PA_2) were derivable from his system including BLV.

On this view:

1. Numbers are *objects* (in particular the number n can be represented by $\{x|$"x has n-many members"$\}$.

2. The laws of arithmetic (PA_2 for Frege) are *part of logic*.

Unfortunately, Frege's system is laid low by Russell's Paradox, after you derive Naive Comprehension (that every condition determines an extension) the condition $x \notin x$ gives you an inconsistency.

> **Side Note.** Russell's attempted solution was to argue that logic should respect *typing* into individuals (type 0), classes of individuals (type 1), classes of classes of individuals (type 2), and so on.
>
> However, the Simple Theory of Types is problematic for several reasons (e.g. meaningful conditions apply to more than one type, the doctrine looks self-undermining, we seem to want to make cross-type comparisons in comparing cardinality, the need for non-logical axioms).

If you'd like to hear more *ask* about this in the discussion period.

This week, we'll examine some contemporary attempts to recover some intuitions from Frege and Russell.

The idea will be the following: While full-blown Logicism might be too strong, the idea that arithmetical truths are *analytic* (or perhaps close to analytic) and can be obtained from *abstraction principles* is still on the right track.

Neo-Logicists give up the idea that we should derive mathematics from logic *simpliciter*.

Rather we should view mathematics in the following terms:

1. We *should* think of numbers as *objects*.
2. They are *characterised* by *abstraction principles*.
3. Some of these abstraction principles are *analytic* (or close to); i.e. they can be viewed as implicit definitions.
4. Our *knowledge* of numbers is thus given by *logic*; they are sui generis objects and there are analytical truths about them.

Frege was thus right (according to the Neo-Logicist) that the objects of arithmetic (and quite possibly other areas) could be understood via logical means, we just have to be a little careful in articulating this claim.

2 Abstraction Principles

First, we need to examine *abstraction principles*.

It is not clear that there is a unique kind of principle denoted by the term "abstraction principle" (it may be more of a family resemblance concept[1]).

We therefore just need to be a touch careful when discussing this notion, considering principles on their own merits, rather than getting too hung up on whether this or that principle deserves the term 'abstraction principle'.

Recall that *abstraction* concerns the idea that we ignore certain specific properties of objects in order to get at something more general.

e.g. I can talk about a line in a space, but forgetting its specific position and length etc., I can talk about its *direction*.

An *abstraction principle* then comes up with criteria of identity between objects on the basis of some similarity between them.

These kinds of similarities are understood via *equivalence relations*.

These are binary relations that are:

(i) **Symmetric:** if bRa then aRb.

[1]See, for example, the discussion of the notion *game* in §66 and §67 of Wittgenstein's *Philosophical Investigations*.

(ii) **Reflexive:** aRa for any a.

(iii) **Transitive:** If aRb and bRc, then aRc.

Some examples: identity, sameness of height, sameness of cardinality.

An *abstraction principle* can be viewed as a way of linking properties of objects under some equivalence relation to more abstract entities.

Basic Law V is one such example!

Another example:

Abstraction for Lines. (In Euclidean geometry) $d(l_1) = d(l_2) \leftrightarrow l_1 \| l_2$ (The direction of l_1 is equal to the direction of l_2 if and only if they are parallel.)

The Neo-Logicist wants to find an abstraction principle for arithmetic.

But they already have one in Hume's Principle!

Definition. *Hume's Principle* (HP) is the following statement (given an operation $\sharp$ that maps concepts to objects):

$$\sharp x F(x) = \sharp x G(x) \leftrightarrow F \approx G$$

i.e. The number of Fs is the same as the number of Gs iff there is a bijection between the Fs and the Gs.

They then claim that Hume's Principle is *analytic* of the concept *cardinal number*; it is *logically true* and *analytic* that the numbers obey Hume's Principle.

If this is the case, we can use Frege's Theorem:

Theorem. (Frege's Theorem) The usual axioms of second-order logic plus HP suffice to derive PA_2.

...thereby arguing that given our concept of number, we can know the theorems of arithmetic (well, PA_2 at least) on the basis of logic and our concept of number.

3 Assessment

There's a lot to be said about Neo-Logicism, and work is ongoing (some of it gets quite technical).

Good Point 1. We can still do the bootstrapping trick (sometimes referred to as Frege's Trick), since Hume's Principle (as interpreted by the Neo-Logicist) introduces numbers as new entities.

The operation $\sharp$ maps concepts to *first-order* objects of the domain.

So we can still get 0 by considering the concept defined by $x \neq x$.

We get 1 by considering the concept $x = 0$.

We get 2 by considering the concept $(x = 0 \vee x = 1)$

We get 3 by considering the concept $(x = 0 \vee x = 1 \vee x = 2)$

...and so on...

The difference with Frege's version is that he could *define* the number n as $\{\{x|F(x)\}|$"F has n-many instances" $\}$

Here we just assume that they exist and are governed by HP.

Remember: The bootstrapping argument was blocked under the Simple Type Theory approach, and we had to use an axiom of infinitely many individuals to get it to work.

Moreover, the typing resulting in the fact that we had many 0s 1s, ..., ns,...

Good Point 2. Since HP is true on some infinite domains, the technical theory given is consistent iff PA_2 is.

Some problems for the view:

Problem 1. *The Julius Caesar Problem.* What is the truth value of $\sharp xF(x) =$ Julius Caesar (or some other random physical object)?

Hume's Principle appears to have nothing to say here, despite the intuitive falsity of the claim.

Problem 2. Is Hume's Principle really analytic and/or logical?

Some points against:

Sub-Problem 2.1. Hume's Principle is only true on infinite domains, but analytic truths are those which hold on any domain.

Response. (Wright) Analytic truths are what hold given logic *and* definitions.

There are infinitely many objects *given* the definition of Hume's Principle as the status of a definition.

This is just the Logicist contention; so to deny it amounts to question-begging.

Sub-Problem 2.2. Hume's Principle has some *staggering* commitments.

It's purview is not just arithmetic, but cardinals more widely.

For example, 0 is still defined via $\sharp x(x \neq x)$.

But what then about *the universal number* or *anti-zero*: $\sharp x(x = x)$?

All things, taken together, have a number!

This is a very strong claim, and perhaps counts against its analyticity.

Two options:

1. Bite the bullet. It's analytic that there is a universal number. The hard work is then making that palatable.
2. Argue that not all predicates define the right kind of concept for being numbered. You then need either a different account of zero (or explain why the one you have is still OK), and a principled account of what differentiates the good from the bad.

Problem 3. Bad company

In general there's the question of how to sort the good abstraction principles (e.g. HP) from the bad ones (e.g. BLV).

But there are abstraction principles that are *individually consistent* but *jointly inconsistent*.

Let $\Delta(F, G)$ abbreviate the claim that finitely many objects are either F-and-not-G or G-and-not-F.

$\Delta(F, G)$ is a perfectly good equivalence relation.

Nuisance Principle. $(\forall F)(\forall G)[\nu(F) = \nu(G) \leftrightarrow \Delta(F, G)]$

Fact. The Nuisance Principle is only satisfiable on *finite* domains.

What happens if we try to put Hume's Principle and the Nuisance Principle together?

Debate is ongoing, and there's a lot of new work of both a philosophical and technical character.

Neo-Logicist ideas have also been extended to parts of analysis and set theory.

It's thus a live position, though unclear exactly how it should be filled out.

4 Questions/Discussion

Question. (Katharine) How appealing is the Fregean/neo-Fregean approach?

Question. (Katharine) What is stopping new axioms from throwing off the calculations made? I.e. Frege ended up with different axioms from Dedekind and Peano.

Question. (Katharine) Is Hume's Principle actually analytic?

Question. (Nicola) Are either natural numbers or cardinal numbers conceptually prior? And does it matter for the neo-logicist?

Question. (Julius/Nicola) Does an abstraction principle require *existence* in order to be true/useful?

Question. (Brian) Is the Caesar problem an issue for the Neo-Fregeans?

Question. More generally, are abstraction principles just ways of *recarving content* (as the Neo-Fregeans would say)?

Question. One purported solution for avoiding the universal number is to say that we never quantify over all objects. There is a universal number for every domain, but not for all domains. What do we think of this view?

Question. One way of viewing the above problem is to think of Hume's Principle as *dynamic* and *modal* (you always introduce new objects). What do we think of the idea of modality in mathematics?