Week 11. Structuralism

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Recap and this week

Last week we discussed the iterative conception of set.

This held that sets are formed in stages, and (if we assume that enough stages exist) could motivate the axioms of ZFC.

This week we'll discuss *structuralism*, the idea that mathematics is concerned with structure.

There's *a lot* of literature here.

I want to leave you with two points:

1. Structure is very important in mathematics, and mostly mathematicians only care about structure.

2. There is a kind of worry that plagues structuralism concerning how it makes sense of its core concepts—the natural way to make sense of structure is just to rely on *more mathematics*.

1 Motivations

Here's a motivation: Mostly in mathematics, we only care about the relationships objects have to other objects, and not about their internal constitution.

Example 1. Does it really matter if we code the ordered pair $\langle a, b\rangle$ as the Kuratowski ordered pair $\{\{a\}, \{a, b\}\}$ or the Hausdorff ordered pair $\{\{a, 0\}, \{b, 1\}\}$, or just have a sui generis notion (with axioms added)?

All these representations have the properties we need for them to fulfil the function of *being an ordered pair*.

Example 2. (Benacerraf) Consider the finite von Neumann ordinals.

$0 =_{df} \emptyset$

$n + 1 =_{df} n \cup \{n\}$

Here, the $n < m$ relation can be represented by $n \in m$.

Now consider the Zermelo finite ordinals:

$0 =_{df} \emptyset$

$n + 1 =_{df} \{n\}$

Here, the $n < m$ relation can be represented by $n \in TC(m)$ ($TC(x)$ is the *transitive closure* of x, just throw in all members, all members of members, etc.)

It doesn't really matter for number-theoretic purposes what we pick as our interpretation of the natural numbers under the less-than relation.

Structuralism. Mathematics is concerned with *structure/pattern* and *not* what the objects *are*.

Note. A nice feature of structuralism: We have the beginnings of a response to the access problem.

We don't need to access specific mathematical objects, we can just *describe* structures using our mathematical theories.

It *doesn't matter* if we refer to this or that object, any object(s) fulfilling the required roles with do.

But now we've got some questions:

1. What are structures?
2. What kinds of structures are there?
3. When are two structures the same?

2 Kinds of structuralism

2.1 Set-theoretic structuralism

Structures are particular kinds of *set* (specifically ordered n-tuples).

There are exactly the structures set theory says there are.

Two structures are the same if they are isomorphic.

Isomorphisms. Given two $\mathscr{L}$ structures $\mathfrak{M}$ and $\mathfrak{M}'$ with domains M and M', an isomorphism $f : M \to M'$ is a bijection such that:

(i) f preserves constants (i.e. $f(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$ sends the interpretation of every constant c in $\mathfrak{M}$ to the interpretation of c in $\mathfrak{M}'$)

(ii) f preserves functions, i.e. $f(h^{\mathfrak{M}}(a_1, .., a_n) = h^{\mathfrak{M}'}(f(a_1), ..., f(a_n))$

(iii) f preserves relations, i.e. $\langle a_1, .., a_n \rangle \in R^{\mathfrak{M}}$ iff $\langle f(a_1), .., f(a_n) \rangle \in R^{\mathfrak{M}'}$

We can represent structures by (restricted) equivalence classes under isomorphism.

Problem. Set theory is just another kind of mathematics.

Shouldn't this be given a structural treatment too?

2.2 Ante rem structuralism

A different option: Assert that structures are a kind of sui generis mathematical entity.

What kinds of structures are there?

For this we need some sort of 'axioms for structures'.

This has been done by e.g. Shapiro in *Philosophy of Mathematics: Structure and Ontology*.

(Roughly speaking, he mimics the axioms of ZFC but in a structural language.)

Problem. When are two structures the same?

Here, we're likely to say again *isomorphism*.

(Perhaps though you want a different equivalence relation, but let's go with this for now.)

But making sense of this requires *mathematics* (usually set theory).

How do we account for *this* mathematics?

2.3 In re structuralism

OK so we got into trouble above.

Instead, rather than say we have some theory of abstract existing structures mediated by isomorphism, let's say that a structure exists just in case there's some system of objects exemplifying it.

What structures exist?

Probably we don't know.

This can have some bad problems, like the *vacuity problem*.

A solution (e.g. Hellman) go *modal*.

Mathematical truth should be understood as claims about logically possible structures and the logically possible functions between them (which in turn, can be thought of structurally).

Problem. How should we understand this modality?

One popular way is to use a model theory (e.g. a Kripke frame) with a set of worlds.

Problem. A lot of mathematics can be encoded modally.

How should we think of this model theory/modal logic?

Is this just more *mathematics*?

Note: There is a really large literature on trying to respond to these kinds of problem.

3 Varieties of structures

One closing point.

For the sake of argument and making the positions clear, I've just considered (set-theoretic) isomorphism as sameness of structure.

However, there is a range of fine-ness of grain and different equivalence relations we could pick.

e.g. The *first-order* structure given by PA, the *second-order* structure given by PA_2.

The latter has a unique model up to isomorphism (given a full semantics) the former does not.

How should we think of these different kinds of structure (whether or not we are 'structuralists')?

4 Questions/Discussion

Cluster: Benacerraf's argument.

Question. (Nicola/Birgit/Haochong) Does Benacerraf's generalisation from sets to any objects whatsoever work?

Cluster: The access problem.

Question. (Brian/Nicola/Michel) How does structuralism handle the access problem, and does it work?

Question. (Birgit) How about for infinite structures?

Cluster: Structures under platonism.

Question. (Fartein) How should we think of talk of structure as a platonist?