Week 10. The Iterative Conception of Set

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Recap

Last week we talked about *neo-Logicism*, the idea that mathematics could be founded on *abstraction principles* that *avoided* Russell's Paradox.

This week we'll examine a different way of prohibiting Russell's paradox, based on the idea of the *iterative conception of set* and ZFC set theory.

1 Why set theory?

There's several reasons to be interested in set theory. (For more, see Penelope Maddy's 'What Do We Want a Foundation To Do?' (2019)).

Here's my top three philosophical reasons:

1. It provides our current best theory of **Infinite Size**.

2. It provides a **Generous Arena** in which all of mathematics can be encoded.

3. It provides (in the wake of Gödel's Theorems) a theory for *calibrating strength* (call this **Metamathematical Corral**).

Let's see how these shake down.

The key point to take away is that set theory is *tremendously powerful* for *encoding* mathematics.

General strategy: Take some mathematical object Y that we talk about in some (possibly very rich) language $\mathscr{L}$.

Find a *set* x and a *translation* of $\mathscr{L}$ into the language of set theory $\mathscr{L}_\in$ (whose sole non-logical symbol is membership $\in$), such that x under the relevant translation in $\mathscr{L}_\in$ has all the properties of Y in $\mathscr{L}$.

1. Infinite size

Recall that we want to say:

X and Y have the *same cardinality* iff there is a bijection $f : X \to Y$.

We first need the definition of *ordered pair*.

The ordered pair $\langle x, y \rangle$ can be represented by the Kuratowski definition $\{\{x\}, \{x, y\}\}$.

(This easily generalises to ordered triples, quadruples, n-tuples etc.)

Functions and relations can then be represented by their *graphs* (sets of ordered pairs).

$f : a \to b =_{df} \{\langle x, y \rangle | f(x) = y\}$

$R =_{df} \{\langle x, y \rangle | xRy\}$

We now need the notion of an *ordinal number*.

Sets X and Y under well-ordering relations[1] R and S respectively have the *same ordinal number* iff there is an *order isomorphism* between (X, R) and (Y, S) i.e. a bijection such that $x_0 R x_1$ iff $f(x_0) S f(x_1)$

We can represent ordinal numbers by particular sets, specifically transitive sets (all members of members are members) well-ordered by the membership relation.

The least ordinal $0 = \emptyset$, $\alpha + 1 = \alpha \cup \{\alpha\}$, limit $\lambda = \bigcup_{\beta<\lambda} \beta$.

Assuming every set can be well-ordered, the *cardinal number* of a set x can then be represented by the least ordinal bijective with x.

One can then define cardinal and ordinal addition, multiplication, and exponentiation.

Theorem. (Cantor's Theorem) For any set x the set of all subsets of x (the power set of x or $\mathcal{P}(x)$) has more elements than x.

Proof. Suppose there is such a bijection $f : x \to \mathcal{P}(x)$. Consider $\{y | y \notin f(y)\}$. Since f is a bijection, $\{y | y \notin f(y)\}$ is hit by f on some $z \in x$. But then $z \in f(z)$ iff $z \notin f(z)$. Contradiction!

We then have a whole hierarchy of infinite sizes that we can do arithmetic with!

Set theory provides our *best theory* of infinite size.

2. Generous Arena

With this in hand, we can also represent many of our favourite mathematical structures.

Natural numbers are represented by the finite ordinals.

$+$, $\times$, $<$ etc. are represented by the relevant relations.

Rational numbers can be represented by pairs of natural numbers (so $\frac{m}{n}$ is represented by $\langle m, n \rangle$) and functions/relations on them can be done similarly.

Real numbers can then be represented by Dedekind cuts, partitions of the rationals into everything to the 'left' of the real number, and everything to the 'right' of the real number.

We can then have $\times$, $+$, $<$ as functions/relations.

Analysis can be represented as functions from reals to reals.

Closely linked is the idea of:

Shared Standard. Provide a standard of correctness for proof in mathematics.

If there's a disagreement about some proof, we can go trace it in the sets.

This is *controversial*, since often the representations are *ugly*.

3. Metamathematical corral

We saw in previous weeks that given Gödel's Theorem, it's hard to tell when a theory is consistent.

[1] R is a *well-ordering* on X iff R is a total order on X and every non-empty subset of X has an R-least element.

This ability to represent mathematics in set theory gives us a way of at least *calibrating* how strong a theory is.

For some new theory you give to me, I can try to find a model for it in the sets.

I then know that the theory is consistent just in case the relevant fragment of set theory I used is.

(We'll see this feature in more detail in Week 12.)

2 The theory ZFC

As we know, Frege's theory of extensions suffered from *Russell's Paradox*:

Naive Comprehension. Let $\phi(y)$ be any condition (but the language of set theory $\mathscr{L}_{\in}$ suffices for the paradox). Then:

$$\exists x \forall y(y \in x \leftrightarrow \phi(y))$$

By using $y \notin y$ (along with basic logic) we can derive a contradiction from Naive Comprehension.

In the early 20th century, the following system came to be used:

ZFC consists of the following axioms:

Axiom of Extensionality. $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$.

Intuitive characterisation. Any two sets with the same members are identical.

Axiom of Pairing. $\forall x \forall y \exists p \forall z[z \in p \leftrightarrow (z = x \vee z = y)]$.

Intuitive characterisation. For any two sets x and y there is a set containing just x and y.

Axiom of Union. $\forall x \exists y \forall z \forall w[(w \in z \wedge z \in x) \rightarrow w \in y]$.

Intuitive characterisation. For any set x, there is a set of all elements of members of x.

Axiom of Choice. If $\mathcal{F}$ is a set of pairwise-disjoint non-empty sets then:

$\exists c \forall x \in \mathcal{F}\, \exists y(c \cap x = \{y\})$.

Intuitive characterisation. For any non-empty set of pairwise-disjoint non-empty sets, there is a set that picks one member from each.

Axiom of Infinity. $\exists x[\exists y\, y \in x \wedge (\forall z\, z \in x \rightarrow z \cup \{z\} \in x)$.

Intuitive characterisation. There is a non-empty set such that if it contains a set z, it also contains z unioned with its singleton. The axiom thus guarantees the existence of an infinite set.

Power Set Axiom. $\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x)$.

Intuitive characterisation. For any set x, there is a set of all subsets of x.

Axiom of Foundation. $\forall x(x \neq \emptyset \rightarrow \exists y \in x\, y \cap x = \emptyset)$.

Intuitive characterisation. Every set contains an element that is disjoint from it. The axiom both rules out self-membered sets and also the existence of infinite descending membership chains.

Axiom Scheme of Separation. If ϕ is a formula in $\mathscr{L}_{\in}$ with y not free then:

$\forall x \exists y \forall z[z \in y \leftrightarrow (z \in x \wedge \phi(z))]$

Intuitive characterisation. Given a set x, one can 'separate' out the ϕs from x into a new set y.

Separation is derivable from the following axiom (*modulo* the other axioms of **ZFC**):

Axiom Scheme of Replacement. Let $\phi(p, q)$ define a function, in the sense that if $\phi(x, y)$ and $\phi(x, z)$ both hold then $y = z$. Then:

$\forall x \exists y \forall z[z \in y \leftrightarrow \exists p \in x \phi(p, z)]$.

Intuitive characterisation. If ϕ defines a function, then the image of any particular set under ϕ is also a set.

Remark. ZFC suffices for all the constructions we discussed above, and is (as far as we know) consistent.

3 The iterative conception

Sets are formed in *stages*. For each stage S there are certain stages which are before S. At each stage S, each collection consisting of sets formed at stages before S is formed into a set. There are no sets other than the sets which are formed at the stages. (Shoenfield, 'Axioms of Set Theory, 1977 p. 323)

The idea is the following:

We start at stage 0 with some objects (this could be nothing, and generally this is all we need)...

...we then take all possible sets we could form....

...and collect them together at limit stages...

...and continue for as far as possible.

We can naturally motivate ZFC on the basis of the iterative conception.

George Boolos, in his article 'The Iterative Conception of Set' provides a formal theory of *stages* (using primitives for *sets*, *stages*, *membership*, *earlier than*, and *is formed at*).

He then shows how ZFC can be derived from various principles about the stages (for a very clean presentation of this idea, see Tim Button's 'Level Theory Part I').

However also ZFC has the iterative conception at it's *heart*.

Within ZFC one can define:

Definition. The cumulative hierarchy is defined as:

$V_0 = \emptyset$

$V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha)$

(This can be reduced to $\mathcal{P}(V_\alpha)$ if we assume all sets are pure—i.e. built up from the empty set.)

For a limit ordinal λ:

$V_\lambda = \bigcup_{\beta<\lambda} V_\beta$

One can then prove:

Theorem. ZFC $\vdash \forall x \exists \alpha x \in V_\alpha$ (i.e. every set belongs to some V_α).

4 Some challenges

Some challenges for ZFC and the iterative conception:

1. Excessive ontology.

ZFC proves that there are *a lot* of objects.

Most of the representative work we want to do is done in the first few levels above V_ω.

But ZFC postulates that there are *huge* objects beyond this.

Do we really *need* these objects? (cf. Boolos 'Must We Believe in Set Theory?', 2000).

2. Proper classes and the paradoxes.

There's a sense in which non-set-like collections are still there.

I can still talk about and compare classes.

e.g. "The universal class and the Russell class are the same."

The membership of these 'collections' is perfectly determinate.

Many (e.g. Linnebo 'Pluralities and Sets', 2010) think that it's arbitrary to insist that they don't exist.

3. The temporal metaphor.

Temporal metaphor at the heart of the iterative conception has been seen by some as problematic.

Really, for a platonist, there is no 'time' etc. this is just colourful talk about the sets.

But this can be leveraged in service of the proper class problem (as in Linnebo).

The 'proper classes' of one stage are sets in the next stage.

For Linnebo, we should just accept that there is no definite sequence of all stages and take the modality seriously.

4. Choice and Replacement.

Two axioms are especially problematic.

Choice. It's hard to see how to get choice out of the iterative conception without just assuming it about the stages.

But it's also clear that it meshes *really well* with the *idea* behind the iterative conception.

Replacement. For replacement, one has to assume it about the stages to get it about the sets.

However, it's also hard to see why it should be true of the iterative conception (see Michael Potter's work here).

However, it can be proven from ideas about *reflection*.

$$\phi \to \exists \alpha \phi^{V_\alpha}$$

Intuitive Characterisation. If ϕ is true, then it is true in some V_α.

This can be motivated from the idea that the universe of set theory should be 'rich', to iterate for 'as long as possible' we should get initial segments of the universe that resemble the universe.

5. Do we isolate a unique structure?

Note that the iterative conception has the powerset operation at its core.

Any indefiniteness in the powerset operation is going to extend to indefiniteness about the hierarchy in general.

But there is a question here out of what is needed from set theory and the iterative conception.

How are the 'nice' features of set theory identified right at the start affected if we lose determinateness?

5 Questions/Discussion

Question. (Ingvild) Should we think of infinity as actual or potential?

Question. (Haochong/Brian) In Parsons' reading it mentions Wang's approach to set formation which involves the overview of (infinite) multitude in an idealized sense, and I wonder whether this idealised overviewing would make Wang an actualist or potentialist?

Question. (Brian/Pietro) What do we think of the justification for AC under the iterative conception?

Question. (Nicola/Haochong) The empty set is an odd object, and at least in pure set theory is needed. Is it ontologically respectable?

Question. (Fartein) Does set theory necessitate platonism?

Question. (Julius) Does the modal move solve the paradoxes?