Week 3. Formalism and Deductivism

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Recap

Last week we discussed Frege and his *logicism*.

This was beset by Russell's paradox.

This week we'll look at a family of views that comes *very close* to the idea that mathematics is part of logic.

There's lots to discuss here, and Haochong will introduce some issues. I will focus on the proposed epistemic payoffs and issue of consistency.

Outline.

§1 The different kinds of formalism/deductivism

§2 The epistemic 'payoff'

§3 The problem of consistency

§4 Discussion

1 The different kinds of formalism/deductivism

Game formalism holds that mathematics is a game played with symbols.

We have *choice* over what rules the game will have.

Symbols have *no* semantic meaning.

Term formalism holds that the semantic denotation of the numerals are the *numerals themselves*.

Mathematics is understood through the *rewrite rules* for the relevant terms and the relevant theory.

We are *free* to pick whatever rewrite rules we want.

Deductivism/if-then-ism holds that mathematics is the study of what follows from what (WFFW), where WFFW is understood via first-order logic.

We are *free* to pick the axioms we want.

We are *not* free to pick the rules we want.

Question. How is **deductivism** different from **logicism**?

Clarifications?

2 The epistemic 'payoff'

Let's first note that game/term formalism are *not* what mathematics *seems* like (at least for many mathematicians, this is discussed by Frege and others).

Part of what these views are claiming to do is give us *epistemic* traction.

A more platonist attitude struggles to account for how we have access to mathematical objects.

Game formalism can hold that we understand mathematics by understanding the *rules of the game* and the *inscriptions* we use.

Term formalism can hold that we understand mathematics by understanding the *basic terms* and *rewrite rules*.

Deductivism holds that we understand mathematics by understanding *derivation in first-order logic*.

One *resemblance* between the views is that there is no question of *justification* in each context (beyond first-order logic) we can just lay down the game/rules/axioms in each case.

3 The problem of consistency

There's *lots* to say about these views, but I want to focus on just one.

Consistency problem. How do we know/justify the *consistency* of our mathematical theories?

Why is this such a *problem*?

First-order logic validates the principle of *explosion*:

$$\bot \vdash \phi$$

for any sentence ϕ.

Optional Exercise. Show why the explosion rule is true in first-order logic.

Inconsistent theories are thus *useless* (given classical first-order logic, or indeed intutionistic logic which also validates explosion).

It *seems* like we have an *advantage* if we think that mathematic is *contentful*.

Roughly speaking. A model $\mathcal{M}$ of a theory T in a language $\mathscr{L}_T$ is an interpretation of the (i) domain, (ii) predicate symbols, (iii) function symbols, and (iv) constants in $\mathscr{L}_{\mathscr{T}}$, such that $\mathscr{M} \models \phi$ for every ϕ in T (this is defined recursively).

Roughly:

Soundness. If something can be proved, then it is true on every model.

Completeness. If something is true in every model, then it can be proved.

First-order logic is both *sound* and *complete*.[1]

From soundness we can get:

Model $\Rightarrow$ **Consistent.** If a theory has a model, then it is consistent.

The person who holds that mathematical claims are contentful has an answer to the consistency problem:

Since mathematical claims have semantic content, they are true on a model, and hence are consistent. (e.g. PA is true on $\mathbb{N} = \langle N, 0, 1, +, \times, < \rangle$)

(**Note.** You might well question how satisfactory this response is. In particular, we're stuck with justifying what's true.)

It is unclear how game formalism and term formalism stack up here.

They do not have semantic access in the same sense (game formalism has none, term formalism has greatly weakened).

The rules can be *whatever you want*, and it's not clear that there's anything demarcating a *bad* rule from a *good* one.

Deductivism can be seen as a way to deal with this issue, by fixing the rules of inference as first-order logic.

It also gives a nice answer to issues of *access*.

We can hold that we don't *need* to access *the* structure (of whatever theory).

First-Order Descriptivism. We just write down some first-order axioms, and that can be true of whatever is out there that *happens* to satisfy those axioms.

But how do we know that we have a structure?

Finite structures we can build (in principle).

(**Note:** We should be wary of this kind of claim.)

Putnam contends that for infinite structures, it is enough that there *could* be such a model.

Spelling out these modal claims is no easy task.

One underdeveloped option. Hold that while *mathematics* is about

[1] Beware the use of the term 'complete'! Sometimes it's used with a different sense to talk about axiom systems too, and some axiom systems in FOL are *in*complete in this sense. We'll see this when we talk about the Incompleteness Theorems.

deductivism, we do have some intuition of *models* (even if not specific ones).

Problem. Does this have any *real* advantage over the semantic contentfulness position?

Second underdeveloped option. Make the study of infinite structures *conditional* on the (possible) existence of a model, and adopt some variety of finitism.

4 Discussion

1. (Haochong) Which theory, (term) formalism or deductivism, do you prefer?
2. (Haochong) Do you agree with the concerns raised toward deductivism?
3. (Haochong) Are there any arguments that could refute the concerns against term formalism?
4. (Katharine) How problematic is the issue of consistency? (e.g. Would it be so bad if there was a contradiction in mathematics that was really long?)
5. (Steinar) How to handle exception rules for a formalist/deductivist vs. a Fregean?
6. (Ingvild Elise) What is the difference between *potential* and *actual* existence in mathematics, and can this be put to work in providing models in this context?
7. (Birgit Margrethe) To what extent is *social convention* allowed in mathematics?
8. (Johan Julius) What kinds of *application* are important in mathematics, and what would have fallen if we didn't have certain parts of pure mathematics (e.g. AC)?
9. How important is it that mathematics be *applied* and what are the *constraints* on a good application?
10. How should we weigh the *phenomenology* of mathematics and *naturality of interpretation*? (and can good sense be made of these notions?)
11. (Michel) What's the difference between *foundational* and *structural* axioms?
12. Is there a *sharp line* between mathematics and logic?
13. (Fartein) There were some important questions raised about some metamathematical results, in particular (i) the Incompleteness Theorems, and (ii) the Löwenheim-Skolem Theorems.

We will discuss both later (i) next week, (ii) towards the end of the course.

14. (Jens Ludvig) There were also some important questions raised about predicates vs. sets in foundations. We can discuss this later when we talk about set-theoretic foundations.