Is it restrictive to say that there are uncountable sets?

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Introduction

Are there uncountable sets?

The 'standard' answer (at least post-Cantor) is that there are, and the reason for this is Cantor's Theorem.

Uncountabilism is the position that there are uncountable sets.

Countabilism is the position that every set is countable.

Main question. Is countabilism a **restrictive** position?

Intuition. Doesn't the uncountabilist just straightforwardly claim that there are *more* sets than the countabilist?

It is the objective of this short paper to argue that this claim is not so straightforward. In particular I will argue that:

Main Claim. There is formal analysis of restrictiveness, based on Maddy's analysis in [Maddy, 1998], on which it is the *uncountabilist* and *not* the countabilist that makes restrictive claims.

Here's the outline:

§1 Why you might think countabilism is restrictive

§2 Why this is too fast

§3 Maddy-restrictiveness

§4 Countabilist maximisation

§5 Philosophical analysis

§6 Conclusions

1 Why you might think countabilism is restrictive

Here's a contrast that might help:

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(Set) Finitism is the position that only finite sets exist.

Transfinitism is the position that some infinite sets exist.

Now finitism seems straightforwardly *restrictive*.

The finitist claims that there are *no* infinite sets—some sets that the transfinitist thinks exist, do not in fact exist.

Isn't the same argument clearly true for the uncountablist vs countabilist?

The countabilist claims that uncountable sets don't exist, the uncountabilist claims that they do exist.

So there are some sets that the uncountabilist thinks exist that the countabilist thinks don't exist.

2 Why this is too fast

To start, I'll make the following two assumptions:

Determinacy of natural number. Our conception of the natural numbers is determinate i.e. every claim about the natural numbers is either true or false (and not both) and our concept picks out a unique structure (possibly up to isomorphism).

Determinacy of well-foundedness. Our conception of well-foundedness is determinate (i.e. when we talk about a well-founded relation, it is *really* well-founded).

Here's why the above argument is too fast.

Let's suppose it is at least *consistent* to have theories postulating transfinite/uncountable sets.

What are the *models* of these theories like under each view?

(**Note:** We'll keep it vague at this stage, and go into more detail below.)

Assume *finitism*. Then there are no really 'nice' models containing transfinite sets.

For them, membership witnesses the finitude of every set.

Any model (of some reasonable fragment of set theory) on which every set is finite is going to be able to see that its sets are finite.

OK, so now assume *countabilism*.

We can still have very nice looking models M that have M-uncountable sets in them.

These models can have the 'real' $\in$-relation.

Here's the **main point.**

If we're uncountabilists then we think that the countabilist lives in some small transitive model containing only countable sets (e.g. $H(\omega_1)$ is a nice choice).

If we're countabilists, we think that the *uncountabilist* lives in models that *miss* out a bunch of subsets (namely the collapsing functions onto ω).

But there's nothing I've said yet to tell between these two situations.

Contrast that with finitism: The uncountabilist thinks that the finitist lives in V_ω.

The finitist agrees, they just think that nothing lives outside of V_ω and everything the transfinitist says should be interpreted as some wholly gerrymandered model.

(**Note:** This disanalogy disappears if drop the **determinacy of natural number**.)

3 Maddy-restrictiveness

OK: That's all well and good, but can we be more specific?

We'll consider a version of *restrictiveness* proposed by Penelope Maddy.

Her account uses the following definitions:

Definition 1. $\mathbf{T}$ *shows* ϕ *is an inner model* iff:

(i) For all $\sigma \in \mathbf{ZFC}$, $\mathbf{T} \vdash \sigma^{\phi}$.

(ii) $\mathbf{T} \vdash \forall\alpha\phi(\alpha)$ or $\mathbf{T} \vdash (\exists\kappa\text{“}\kappa \text{ is inaccessible”} \wedge \forall\alpha[\alpha < \kappa \rightarrow \phi(\alpha)])$

(iii) $\mathbf{T} \vdash \forall x \forall y([x \in y \wedge \phi(y)] \rightarrow \phi(x))$.

Definition 2. ϕ is a *fair interpretation of* $\mathbf{T}_1$ *in* $\mathbf{T}_2$ (where $\mathbf{T}_1$ extends $\mathbf{ZFC}$) iff:

(i) $\mathbf{T}_1$ shows ϕ is an inner model, and

(ii) for all $\sigma \in \mathbf{T}_1$, $\mathbf{T}_2 \vdash \sigma^{\phi}$.

Definition 3. $\mathbf{T}_2$ *maximizes* over $\mathbf{T}_1$ iff there is a ϕ such that:

(i) ϕ is a fair interpretation of $\mathbf{T}_1$ in $\mathbf{T}_2$.

(ii) $\mathbf{T}_2 \vdash \exists x \neg\phi(x)$.[1]

Definition 4. $\mathbf{T}_2$ *properly maximizes over* $\mathbf{T}_1$ iff $\mathbf{T}_2$ maximizes over $\mathbf{T}_1$ but $\mathbf{T}_1$ does not maximize over $\mathbf{T}_2$.

Definition 5. $\mathbf{T}_2$ *inconsistently maximizes over* $\mathbf{T}_1$ iff $\mathbf{T}_2$ properly maximizes over $\mathbf{T}_1$ and $\mathbf{T}_2$ is inconsistent with $\mathbf{T}_1$.

Definition 6. $\mathbf{T}_2$ *strongly maximizes over* $\mathbf{T}_1$ iff $\mathbf{T}_2$ inconsistently maximizes over $\mathbf{T}_1$ and there is no consistent extension of $\mathbf{T}_1$ that properly maximizes over $\mathbf{T}_2$.

Definition 7. $\mathbf{T}_1$ is *restrictive* iff there is a consistent $\mathbf{T}_2$ that strongly maximizes over $\mathbf{T}_1$.

The rough idea of Maddy's proposal is that a theory $\mathbf{T}_2$ extending $\mathbf{ZFC}$ is maximises over another $\mathbf{T}_1$ just in case:

1. $\mathbf{T}_2$ is consistent,
2. $\mathbf{T}_2$ inconsistent with $\mathbf{T}_1$,
3. $\mathbf{T}_2$ can represent $\mathbf{T}_1$ in an appropriately 'nice' context (either an inner model, truncation at an inaccessible, or an inner model of a truncation at an inaccessible), and

[1] Maddy actually has a slightly more complicated definition, but (as she) notes, this condition suffices for current purposes in the presence of Foundation, and all our theories have (on account of being iteratively grounded).

4. There's no way of extending $\mathbf{T}_1$ to get a 'nice' context.

For example, Maddy takes this to show that $V = L$ is restrictive since **ZFC**+"There exists a measurable cardinal" is inconsistent with $V = L$, has an inner model satisfying $V = L$ (namely L), and L cannot produce a nice model for **ZFC**+"There exists a measurable cardinal".

4 Countabilist maximisation

We want to consider cases where we don't have **ZFC** as the base theory, but rather allow every set to be countable.

We first need to settle some countabilist theories, since formulating them isn't trivial:

Definition 8. We distinguish between the following theories:

(1.) $\mathbf{ZFC}-$ is **ZFC** with the Powerset Axiom Removed and AC formulated as the claim that every set can be well-ordered.

(2.) $\mathbf{ZFC}^-$ is $\mathbf{ZFC}-$ with the Collection and Separation Schema substituted for the Replacement Scheme.

(3.) $\mathbf{ZFC}^-_{Ref}$ is $\mathbf{ZFC}^-$ with the following schematic reflection principle added (for any ϕ in the language of set theory):

$$\forall x \exists A (x \in A \wedge \text{"}A \text{ is transitive"} \wedge \phi \leftrightarrow \phi^A)$$

i.e. for any set x there is a transitive set A such that $x \in A$ and ϕ is absolute between A and the universe. We will refer to this principle as (the) *First-Order Reflection (Principle)*.

(4.) By $\mathbf{NBG}-$, $\mathbf{NBG}^-$, and $\mathbf{NBG}^-_{Ref}$ we mean the corresponding versions of **NBG**, with two sorts of variables and any corresponding schema replaced by single second-order axioms.

Definition 9. In Maddy's definition, replace every occurrence of **ZFC** with $\mathbf{ZFC}^-$. We say that a theory $\mathbf{T}_2$ extending $\mathbf{ZFC}^-$ *modified Maddy maximizes* over $\mathbf{T}_1$, iff $\mathbf{T}_2$ *strongly maximizes* over $\mathbf{T}_1$ in this new sense (we will simply refer to this phenomenon as strong maximization from hereon out).

Fact 10. *Let* A *be one of the usual large cardinal axioms (of course this is somewhat imprecise, so the reader should feel free to substitute "There is a measurable cardinal" if they wish to have a precise result). The theory* $\mathbf{ZFC}^-$*+"Every set is countable"+"There is a definable inner model for* $\mathbf{ZFC} + \mathsf{A}$*" always strongly maximises over* $\mathbf{ZFC} + \mathsf{A}$.

This, in particular, implies the following:

Fact 11. $\mathbf{ZFC}^-_{Ref}$ *+ "Every set is countable" + Projective Determinacy strongly maximizes over* **ZFC** *+ "There are n-many Woodin cardinals" (pick your favourite n).*

PD isn't an *especially* natural axiom though (though nor are Woodin cardinals).

Can we do better?

Definition 12. *Ordinal Inner Model Hypotheses.* [Barton and Friedman, S] The *Ordinal Inner Model Hypothesis for* $\mathbf{T}$ or $\mathsf{OIMH}^{\mathbf{T}}$ states that if a first-order sentence $\phi(\vec{a})$ with **ordinal** parameters $\vec{a}$ in V is true in a definable inner model $I^* \models \mathbf{T}$ of an outer model $V^* \models \mathbf{T}$ of V obtained by a definable pretame class forcing, then $\phi(\vec{a})$ is already true in a definable inner model $I \models \mathbf{T}$ of V. We shall use OIMH^- and OIMH^-_{Ref} to denote the OIMH for $\mathbf{ZFC}^-$ and $\mathbf{ZFC}^-_{Ref}$ respectively.

Note. This 'axiom' is **not** first-order expressible.

We can't express that something is a model of $\mathbf{ZFC}^-$ or $\mathbf{ZFC}^-_{Ref}$ with one sentence (in the powerset context, the V_α hierarchy handles this issue).

We are now quantifying over models existentially, so the conclusion is an infinite disjunction.

The natural formulation of the OIMH^- and OIMH^-_{Ref} are thus not first-order and is not even given by a first-order scheme (i.e. infinite conjunction of first-order sentences).

Instead it is an infinitary Boolean combination of first-order sentences of low infinitary rank.

Note 2. All this goes away if you adopt $\mathbf{MK}^-_{Ref}$ (modifying the results of [Antos et al., 2021]).

Theorem 13. *[Barton and Friedman, S]* $\mathbf{ZFC}^-_{Ref}$ + OIMH^-_{Ref} *is consistent relative to the theory* $\mathbf{ZFC}$ + PD.

Theorem 14. *[Barton and Friedman, S] Suppose that* V *satisfies* $\mathbf{ZFC}^-_{Ref} + \mathsf{OIMH}^-_{Ref}$*. Then* V *satisfies "*$0^\sharp$ *exists".*

Corollary 15. *The first-order part of* $\mathbf{ZFC}^-_{Ref} + \mathsf{OIMH}^-_{Ref}$ *strongly maximizes over* $\mathbf{ZFC}$*. Indeed the first-order part of* $\mathbf{ZFC}^-_{Ref} + \mathsf{OIMH}^-_{Ref}$ *strongly maximizes over any extension of* $\mathbf{ZFC}$ *with large cardinals we obtain in* L *under the existence of* $0^\sharp$*.*

(**Note:** OK this is a bit naughty— "the first-order part of $\mathbf{ZFC}^-_{Ref} + \mathsf{OIMH}^-_{Ref}$" isn't a recursive theory, and Maddy-maximization is meant to apply to recursive theories. But there are fragments of this theory that are recursive, e.g. $\mathbf{ZFC}^-_{Ref}$ + "$0^\sharp$ exists".)

So, there are countabilist theories that modified-Maddy-maximize over uncountabilist ones.

5 Philosophical analysis

What to say?

In a sense the countabilist perspective of "You miss out subsets" is vindicated over the uncountabilist perspective of "You don't iterate a legitimate operation (powerset)".

However, there are many reactions one might have.

Option 1. Accept that countabilism is correct.

Response. I like this option. *However*, there are then a host of challenges. e.g. how do we understand mathematics more generally under countabilism?

Option 2. Press other concerns against the countabilist (e.g. a lack of iterative conception).

Response. There are ways we might come up with an iterative conception, say by adopting a modal story, or adopting a different hierarchy.

Option 3. Just make your uncountabilist theory strong enough.

Response. Already we are able to get $0^\sharp$.

This is often seen as the limit of 'intrinsic' justification for uncountabilist theories.

I fail to see what could be appealed to here that the countabilist couldn't also make use of.

Option 4. Reject my modification to Maddy-style restrictiveness (e.g. by saying that a fair interpretation is just a countable model.

Comment. This is a good response.

However it does nothing the *vindicate* the uncountabilist.

It just puts them back on equal footing.

There is something *nice* about being able to have all ordinals.

Option 5. Reject Maddy-style restrictiveness altogether.

Comment. This response is fine, as far as it goes.

But it raises the question of what (and if) anything should be put in its place.

6 Conclusions

I think countabilism is not just a live philosophical option, but a live mathematical one too.

This all comes back to the question: **What do we want out of a foundation?**

Do we really **need** a hierarchy of uncountable sets?

Or is it enough to find 'good' interpretations of our mathematical results.

References

[Antos et al., 2021] Antos, C., Barton, N., and Friedman, S.-D. (2021). Universism and extensions of V. *The Review of Symbolic Logic*, 14(1):112–154.

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