Week 15: The Continuum Hypothesis and Our Choice of Axioms

Introduction to the Philosophy of Mathematics
Dr. Neil Barton
neil.barton@uni-konstanz.de

12. February 2020

Recap

Last time we examined the notion of *computability* and *computation*.

We saw how there are mathematical problems that *cannot be decided* by Turing machines and the formal theories they represent.

We also saw, a couple of sessions ago, that there are sentences (e.g. $Con(\mathbf{PA})$) that cannot be decided by the axioms of the relevant axiom system (e.g. $\mathbf{PA}$).

This week we'll merge everything together and see how there are interesting philosophical problems surrouding:

(1.) The kinds of formal theories we use.

(2.) Limitative results.

(3.) The kinds of infinity that exist.

1 One Contemporary Solution to the Paradoxes: The Iterative Conception

Let's first start with the contemporary view infinity.

When we talk about *infinity*, we usually

The idea is the following *Iterative* Conception of Set.

The thought is that sets a formed in *stages*.

I start at stage 0 with nothing.

I then (at stage 1) take all the sets I can form out of things at stage 0, and collect them together (this is just $\emptyset = \{\}$).

I then (at stage 2) take all the sets I can form out of things at stage 1, and collect them together (this is $\{\emptyset, \{\emptyset\}\}$).

I then (at stage 3) take all the sets I can form out of things at stage 2, and collect them together (this is $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$).

...and so on....

I then assume I have an infinite stage, called stage ω. It's just everything I formed at some stage n.

I take all the subsets I can form out of stage ω, to get to stage $\omega + 1$, this is just the set of all subsets (powerset) of stage ω.

At stage $\omega + 2$, I take the powerset of stage $\omega + 1$.... etc.

I then assume I can collect in the limit, to get a stage $\omega.2$.

I take the powerset to get to stage $\omega.2 + 1$...

... and so on ...

This *rough* idea is what lies behind the *iterative conception of set*.

It's *iterative*, because you can think of it as iterating the operations of powerset and union to infinity (and beyond)!

If we work under the iterative conception, notice:

Fact. There is no set of all sets.

We know (by Cantor's Theorem) that the power sets get bigger and bigger. So at no stage do we collect together all the sets, there are always sets at successive stages.

More specifically, for any set x first formed at stage α, its singleton $\{x\}$ is formed at stage $\alpha + 1$.

Fact. There is no Russell set.

Notice that *every* set is non-self-membered on the iterative conception, since if $x = \{x, ..., a, b, ...\}$, and x was formed at stage α, then x would also have to be formed *before* stage α, since that's the only way it could be a member of itself.

So the problem of the Russell class and the problem of the universal class are the same under the iterative conception, and they both aren't sets.

Thus, the iterative conception prohibits the two problematic classes we've looked at from existing.

More generally: The iterative conception (putatively) provides an explanation of *why* Naive Comprehension is false.

Recall that Naive Comprehension states that:

$(\exists a)(\forall x)(x \in a) \leftrightarrow \phi(x)$

(This follows from Frege's system, in slightly different language, and was what got him into trouble.)

But if $\phi(x)$ has satisfiers ever later in the stages, you'll never get a set of all of them.

So, one might think, the iterative conception gives us a reason why Naive Comprehension is false.

Open, Interesting, and Difficult Philosophical Problem. OK, so the Russell class and Universal class aren't sets, nor does an arbitrary condition always define a set. But I still *talk about these entities*! So how should we interpret this talk?

2 The Axioms: ZFC

The iterative conception is a nice idea, but its not yet formal.

We can formalise the idea in **ZFC** (**Z**ermelo-**F**raenkel Set Theory with the Axiom of **C**hoice).

We have:

Axiom of Extensionality.
$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$.

Intuitive characterisation. Any two sets with the same members are identical.

Axiom of Pairing.
$\forall x \forall y \exists p \forall z[z \in p \leftrightarrow (z = x \vee z = y)]$.

Intuitive characterisation. For any two sets x and y there is a set containing just x and y.

Axiom of Union.
$\forall x \exists y \forall z \forall w[(w \in z \wedge z \in x) \rightarrow w \in y]$.

Intuitive characterisation. For any set x, there is a set of all elements of members of x.

Axiom of Infinity.
$\exists x[\exists y\, y \in x \wedge (\forall z\, z \in x \rightarrow z \cup \{z\} \in x)$.

Intuitive characterisation. There is a non-empty set such that if it contains a set z, it also contains z unioned with its singleton. The axiom thus guarantees the existence of an infinite set.[1]

[1] This claim is made slightly more complex by the fact that within set theory there are different definitions of the notion of *infinite set*. This is usually made precise by defining a set to be *finite* iff it is bijective with (a von Neumann representative of) a natural number (and infinite otherwise) or, alternatively, being *Dedekind-infinite* iff it is bijective with a proper subset of itself. The issue is somewhat subtle, as the two notions can come apart in the absence of the Axiom of Choice. See [Jech, 2002], p34 for details.

Power Set Axiom.
$\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x)$.

Intuitive characterisation. For any set x, there is a set of all subsets of x.

Axiom of Foundation.
$\forall x(x \neq \emptyset \rightarrow \exists y \in x\, y \cap x = \emptyset)$.

Intuitive characterisation. Every set contains an element that is disjoint from it. The axiom both rules out self-membered sets and also the existence of infinite descending membership chains.

Axiom Scheme of Separation. If ϕ is a formula in $\mathscr{L}_\in$ with y not free then:

$\forall x \exists y \forall z[z \in y \leftrightarrow (z \in x \wedge \phi(z))]$

Intuitive characterisation. Given a set x, one can 'separate' out the ϕs from x into a new set y.

Axiom Scheme of Replacement. Let $\phi(p, q)$ define a function, in the sense that if $\phi(x, y)$ and $\phi(x, z)$ both hold then $y = z$. Then:

$\forall x \exists y \forall z[z \in y \leftrightarrow \exists p \in x \phi(p, z)]$.

Intuitive characterisation. If ϕ defines a function, then the image of any particular set under ϕ is also a set.

Axiom of Choice. If $\mathcal{F}$ is a set of pairwise-disjoint non-empty sets then:

$\exists c\, \forall x \in \mathcal{F}\, \exists y(c \cap x = \{y\})$.

Intuitive characterisation. For any non-empty set of pairwise-disjoint non-empty sets, there is a set that picks one member from each.[2]

[2] There are a large number of equivalents of the Axiom of Choice, both within set theory and from other areas of mathematics. I chose this formulation be-

Fact. One can find representatives for all the 'usual' objects of mathematics in **ZFC**.

Example 1. Any *ordered pair* $\langle x, y \rangle$ for objects x and y we already have can be coded by $\{\{x\}, \{x, y\}\}$ (this is the *Kuratowski* ordered pair), and we denote it from now on by $\langle x, y \rangle$.

Example 2. A function $f : a \to b$ for objects a and b can be coded as the the set $\{\langle x, y \rangle | f(x) = y\}$ (this is often called the *graph* of f).

Example 3. An *ordinal number* can be defined as a transitive set α (i.e. all members of elements of α are also members of α) well-ordered by the membership relation $\in$.

Example 4. The *cardinal number* of a set x can be interpreted as the least ordinal *bijective* with x. We can then *define* the $\aleph_\alpha$-function indexing the cardinals.

Example 5. We can then define the natural numbers as the smallest infinite ordinal ω, and the members of ω as individual numbers n.

Example 6. You can then define the *rational numbers* as *pairs* of naturals, so $\frac{m}{n}$ is coded by $\langle m, n \rangle$.

Example 7. You can then define the reals in various ways. Given what we've got, *Dedekind cuts* are a natural choice. You can then show that the reals and $\mathcal{P}(\omega)$ are bijective.

Fact. **ZFC** proves $Con(\mathbf{T})$ for *almost* all mathematical theories, since you can find structures in the sets representing these theories.

From within **ZFC** (in fact **ZF**) we can define the following:

Definition. The *cumulative hierarchy of sets* is defined as follows:

$V_0 = \emptyset$

$V_{\alpha+1} = \mathcal{P}(V_\alpha)$ (at successor stages)

At limits λ:

$V_\lambda = \bigcup_{\beta<\lambda} V_\beta$

Fact. **ZFC** $\vdash (\forall x)(\exists \alpha) x \in V_\alpha$

So **ZFC** (in fact **Z**) formally codifies the idea that the sets are obtained by iterating the powerset operation and collecting at limits.

(**N.B.** In the above, I've used various symbols not in $\mathscr{L}_\in$. They can all be treated as defined symbols.)

Question. Why should we believe **ZFC**?

One Answer. It is motivated by the *iterative conception*!

Extensionality, pairing, union, are all fairly clear.

Foundation follows from the fact that the stages are well-ordered (i.e. for any collection of stages, there's a least one).

Separation is motivated by the idea that I take *all* sets at successor stages, so if I can separate out the ϕs I should have a set of them.

Choice is motivated similarly. If I form *all* sets at successor stages, then I should have a choice set, it's like picking one sweet from every jar of

cause it most naturally meshes with motivation from the Iterative Conception of Set.

a (very large) sweet shop.

Replacement is motivated by the idea that I should iterate the stages as *far as possible*, if I can go on to collect the sets into a new stage, I should.

Problem. It's very unclear if these motivations work, or should be epistemically convincing, since I need to assume versions of the principles about the stages in assuming they hold about sets.[3]

Pen Maddy's Response.[4] Let's distinguish between *two* kinds of justification:

Intrinsic justification is when a principle is supported by an intuitive underlying conception.

Extrinsic justification is when a principle is supported by the fact that it gives useful consequences.

Really **ZFC** is *extrinsically* justified (except maybe Extensionality), we believe it because it is so useful in systematising mathematics, and the iterative conception is just a nice post-hoc heuristic for thinking about sets.

Question. Is it clear that intrinsic and extrinsic justification can be neatly separated this way?

[3][Boolos, 1971] gives a formal theory of stages, and shows how it interprets **ZFC**. [Potter, 2004] provides a book-length treatment of what follows from the iterative conception (as well as functioning as an introductory textbook to set theory), but his viewpoint is controversial!

[4]Especially in [Maddy, 2011].

3 The Continuum Hypothesis and the Future

We recall the following theorem:

Theorem. (Cantor's Theorem) For all x, $|\mathcal{P}(x)| > |x|$.

Observation. Assuming **Z** (i.e. not necessarily Replacement or Choice) there is an infinite hierarchy of cardinals.

In particular we know that ω is the smallest infinite cardinal, and that $\mathcal{P}(\omega)$ is larger.

But is there anything *in between*?

The Continuum Hypothesis. There is no cardinal number intermediate in size between ω and $\mathcal{P}(\omega)$. So $\omega = \aleph_0$ and $|\mathcal{P}(\omega)| = |2^{\aleph_0}| = |\mathbb{R}| = \aleph_1$.

Hilbert included proving (or refuting) the continuum hypothesis as number one on his list of problems presented to the International Congress of Mathematicians in 1900.

Fact. (Gödel, 1938) Assuming **ZFC** is consistent, you can build a model (called the *constructible* universe or L), such that $L \models$ **ZFC** + CH, and hence if **ZFC** is consistent, **ZFC** $\nvdash \neg$CH.

Fact. Assuming **ZFC** is consistent, you can build a model $L[G]$ (using *forcing*, by adding subsets of natural numbers) such that $L[G] \models$ **ZFC** + $\neg$CH, and hence if **ZFC** is consistent, **ZFC** $\nvdash$ CH.

Note: CH is *not* like consistency sentences. For example: If you're using **ZFC**, presumably you accept $Con(\mathbf{ZFC})$ (you probably shouldn't use a theory you have serious suspicions is ω-inconsistent if you're trying to represent mathematics). CH is a different matter. I can use **ZFC**, and not have any idea about the truth value of CH.

Two responses to the problem of CH.

Universism. (Gödel) The iterative conception of set determines a perfectly precise universe. So CH is a determinate problem, we just haven't found the axioms yet.

Multiversism. The independence results show that our thought is *indeterminate* and what we really investigate are just *models*. CH does not have a determinate truth-value, and the problem is essentially solved.

There's lots more to be said (in particular I've said nothing about *large cardinals*, which show how 'size' considerations and consistency are linked). We can talk about this in discussion, if you would like.

Key Question. Can we justify new axioms for set theory? If so, how?

4 Questions and Discussion

Fab questions again this week.

Question. Do you think there is a single universe of sets?

Question. If there is/isn't, is the question of justifying new axioms over?

Question. Is one of intrinsic/extrinsic justification more important?

Question. Is there such a thing as mathematical data? If so, how might we acquire it?

Question. What if some set-theoretic axiom had some *physical* consequences? Would that affect its prospects for justification?

Question / Optional Exercise. (Very tricky) Can you think of a *physical* consequence of the truth of the axioms of **ZFC**?

Hint. Think about the relationship between *truth*, *consistency*, and *Turing machines*.

References

[Boolos, 1971] Boolos, G. (1971). The Iterative Conception of Set. *The Journal of Philosophy*, 68(8):215–231.

[Jech, 2002] Jech, T. (2002). *Set Theory*. Springer.

[Maddy, 2011] Maddy, P. (2011). *Defending the Axioms*. Oxford University Press.

[Potter, 2004] Potter, M. (2004). *Set Theory and its Philosophy: A Critical Introduction*. Oxford University Press.