Week 12: Formalism and Finitism

Introduction to the Philosophy of Mathematics
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Recap

Over the last couple of weeks we've examined *intuitionism*; a position that holds that we should be *revisionary* about mathematical practice.

In particular, they think that the Law of Excluded middle and Double Negation Elimination fail.

This week we'll look at the ideas of David Hilbert (1862–1943), who had a position somewhere between the two.

He wanted to eliminate talk about the infinite, whilst being *non-revisionary* about mathematics.

1 Formalism and Finitism

Hilbert's idea was that the paradoxes should be diagnosed as arising from the notion of infinity.

He therefore proposed that only claims about the finite were acceptable. Call this position **finitism**.[1]

Finitism can be presented in at least three different ways:

Ontological: There is nothing infinite (sets, sequences).

Semantic: Only statements that are

(a) about finite entities, and

(b) decidable by finitary means

express real propositions.

Epistemological: Finitary reasoning is secure; non-finitary reasoning introduces uncertainty.

1.1 Ontological Finitism

Hilbert's argument for Ontological Finitism: Nothing infinitely small or infinitely big has ever been found in reality.

So it seems that we cannot think of anything infinite.

But: Isn't part of what we do in mathematics to conceive of alternate models of the way things are? Couldn't

[1] Hilbert scholarship is a little difficult as he seemed to consider more structuralist ideas (the view that mathematics is about structures) in the 1920s. We'll just consider Hilbert the finitist here.

part of this be to conceive of an infinite structure?

Can't we write down things that are only true on infinite structures (e.g. "every natural number has a successor"[2]?

Response. Any time you try and specify any infinite thing you'll always come up short. There seems to be a fundamental clash in intuitions here between the friend of infinite sets and the finitist.

1.2 Semantic Finitism

Semantic finitism arises out of considerations somewhat similar to the intuitionists.

On this view we want to treat statements that are *decidable* as meaningful.

Ideal statements have non-finitary content: They imply or presuppose the existence of infinite entities or are not in principle finitarily decidable.

Given some sort of verificationsim (the idea that only verifiable or falsifiable statements are meaningful), ideal statements are not meaningful.

Contrast: "The 999th digit of π is 9." and "The expansion of π repeats the sequence 47628947497 infinitely often."

The former has a decision procedure, whereas the latter does not.

Much of the plausibility of semantic finitism turns on the verificationist issues we discussed with Dummett, so I won't revisit them here.

Note though that semantic finitism is compatible with using formal theories of infinite sets (thought of as symbols), these ideal statements (even if not meaningful) might be useful for proving facts about real statements, even if you're a semantic finitist.

1.3 Epistemological Finitism

Recall for Epistemological Finitism we have the claim that finitary reasoning is secure, but non-finitary reasoning introduces uncertainty.

Finitary reasoning contains two key parts:

(1.) Fintary subject matter.

(2.) Finitary methods.

What is finitary subject matter for Hilbert?

Answer. Surveyable arrays of clearly perceptible objects with clearly perceptible shapes and spatial relations.

e.g. Hilbert strokes: ||||||

e.g. A formal language (e.g. symbols, strings of symbols), and their formal properties (e.g. being a formula, being a proof).

These properties are perceptually decidable.

Worry. *The type/token distinction.* How many letters are below?

[2] You can write this down in first-order logic as $\forall x \neg R(x,x) \land \forall x \forall y \forall z\ ((R(x,y) \land R(y,z)) \rightarrow R(x,z)) \land \forall x \exists y R(x,y)$.

AA

These are two tokens of the same letter type. But the type is abstract where tokens are not. But Hilbert seems concerned with *types* rather than tokens (e.g. With the claim that $1 \neq 1$ is not a consequence of a particular axiom system.)

Solution. Relax Hilbert's criterion to allow for types whose tokens are clearly perceptible.

Worry. *Practical limits.* Some arrays are unsurveyable (e.g. a formula with more symbols than atoms in the observable universe).

Solution. Talk schematically. Formulas in the metalanguage can be schemas for formulas in the object language (e.g. when we say that an instance of $\phi \to (\psi \to \phi)$ is an axiom for any ϕ and ψ.

What then are finitary methods?

Answer. There is some debate here, but mostly it is agreed that **Primitive Recursive Arithmetic** (**PRA**) is acceptable.

I won't go through the axioms of **PRA**, but roughly speaking it contains:

(i) Principles for defining complex function terms from simple terms in such a way that the value of a function term, for any appropriate input, can be computed (the Primitive Recursive Functions), and symbols for these function terms.

(ii) Propositional axioms (e.g. $\phi \to (\psi \to \psi)$.

(iii) Equality axioms (e.g. for a terms τ, σ, $\tau = \tau$; $\sigma = \tau \to (\phi(\sigma) \to \phi(\tau))$.

(iv) Rules: Substitution, modus ponens, and free-variable induction with decidable predicates (from $\phi(0)$ and $\phi(y) \to \phi(s(y))$ conclude $\phi(y)$).

Note: There are *no* quantifiers here!

You can think of **PRA** as comprising a fragment of arithmetic where all the functions are easily mechanically computable.

Worry. We are prone to make slips in reasoning when manipulating long and/or complex symbol arrays.

Answer. The use of **PRA** assuages this worry; reasoning is mechanically *checkable*, and the formal reasoning does not *depend* upon any particular interpretation of the formal theories.

In this respect, we might think that Epistemological Finitism delineates especially secure forms of reasoning, even if we don't buy finitism in general.

2 Hilbert's Programme

Despite his rejection of the infinite as 'real', Hilbert was nonetheless in many ways positive about our use of talk concerning infinite sets:

> Aus dem Paradies, das Cantor uns geschaffen, soll uns

> niemand vertreiben können. ([Hilbert, 1926], p. 170)

...and indeed, against the intuitionists, he thought that one should be allowed to do so using the law of excluded middle:

> Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists. ([Hilbert, 1927], p. 426)

Why is this? Well, Hilbert was an *instrumentalist*, he thought we could use *ideal* statements in proving facts about *real* ones.

For this he initiated what is now known as **Hilbert's Programme**.

The idea was to show by purely finitary means that our ideal mathematical theories (arithmetic, analysis, and set theory) have no false finitary consequences.

His strategy:

(1.) Set out arithmetic, analysis and set theory as completely precise formal systems.

(2.) Show by finitary means (i.e. in $\mathbf{PRA}$) that no finitary falsehood is derivable in these systems.

The idea was to set out *axiom systems* for infinitary mathematics, and think of our work with ideal mathematics as just computing with these symbols, rather than talking about anything 'real'.

Hilbert hoped to show that these axiom systems, thought of as meaningless symbols, would never produce a contradiction.

Next time, we'll see (with Gödel's Theorems) that Hilbert's Programme cannot be fully realised.

However, the *way* in which it cannot be realised and the *partial* realisations of it have been absolutely fundamental in shaping the foundations of mathematics.

3 Questions / discussion

Question. Is it acceptable to use ideal theories in proving facts about reality?

Question. Is there more content to mathematical subject matter than the formal theories we use?

Question. Does the finiteness (or otherwise) of the physical universe have any bearing on whether or not we should hold mathematical finitism?

Question. Is there an important philosophical difference between being unbounded and being infinite?

Question. How might Hilbert justify basic logical laws?

Question. Why is it so important for Hilbert to defend infinity when he thinks it is not to be found in reality?

Question. Is removing the Law of Excluded Middle tantamount to depriving the boxer the use of his fists (and relinquishing science)?

References

[Hilbert, 1926] Hilbert, D. (1926). Über das unendliche. *Mathematische Annalen*, 95(1):161–190.

[Hilbert, 1927] Hilbert, D. (1927). The foundations of mathematics. In van Heijenoort, J., editor, *From Frege to Gödel: A Source Book in Mathematical Logic*. Harvard University Press.