Week 11: Intuitionism and Meaning: Dummett

Introduction to the Philosophy of Mathematics
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Recap

Last week, we saw a very different attitude to infinity, and in particular Cantor's idea that some infinities could be treated as objects in their own right.

Instead, Brouwer held that:

(i) Mathematical reality comes into being with our acts of *mental construction*.

(ii) There is no distinction between what is true in mathematics and what has been proved.

(iii) There is no untensed truth.

This came with an attendant conception of intuitionistic *logic* and *mathematics* on which:

1. The meaning of the connectives can be interpreted as algorithms for constructing proofs.
2. Double Negation Elimination $\neg\neg\phi \rightarrow \phi$ fails (although DN-Introduction $\phi \rightarrow \neg\neg\phi$ is intuitionistically valid).
3. The Law of Excluded Middle can fail (i.e. it is not necessarily the case that $\phi \vee \neg\phi$ for every ϕ).

1 Some objections to Brouwerian Intuitionism

Let's first finish some issues we didn't quite consider in detail from the last seminar:

Phenomenological Challenge. This talk of bare-two-one-ness is at best perplexing, and at worst incoherent (especially when we want a foundations for *mathematics*, which is meant to be especially secure).

Identity Challenge. Let's grant some kind of interpretation of bare-two-one-ness and mental construction. How many 2s are there? Is my 2 the same as your 2? What happens if we construct *the same* proof? How do we account for that claim?

Temporal Challenge. Wasn't it *always* the case (or not) that there were (or weren't) exactly two helium atoms orbiting the sun? What does any *actual* construction have to do with it?

It seems then that Brouwerian Intuitionism, whilst intriguing, is philosophically problematic.

Michael Dummett (1925–2011) proposed a different way of motivating intuitionism that (whilst still controversial) avoids some of this more problematic aspects of Brouwer's philosophical baggage.

2 Dummett's Intuitionism: Meaning

Dummett's Intutionism arises out of considerations about *language* and *meaning*, rather than on the basis of an underlying *ontological account*.

Dummett is quite precise (once you've read things a few times), but it's a little difficult to untangle what he says (there's lots of moving parts), so let's walk through it slowly. Key is the claim that meaning is prior to questions of ontology:

> ...a philosophical account of thought can be attained through a philosophical account of language, and, secondly, that a comprehensive account can only be so attained. ([Dummett, 1993], p. 4)

Dummett's Meaning Principle. We can only come to an adequate account of (the) philosophy (of mathematics) through considerations of language and what we *mean* by our terms.

Indeed this is how he diagnosed some of the failings of Brouwer's account; he claims that Brouwer failed to take enough notice of linguistic considerations:

> traditional intuitionist accounts, which, notoriously, accord a minimum of importance to language or to symbolism as a means of transmitting thought, ... are constantly disposed to slide in the direction of solipsism. ([Dummett, 1975], p. 17)

Two principles about language are key for Dummett's philosophy, one arising from the work of Frege, and the other Wittgenstein:

The first is Frege's *Context Principle*, made in the *Grundlagen*, and characterised by Dummett as:

> ...the thesis that it is only in the context of a sentence that a word has a meaning: the investigation therefore takes the form of asking how we can fix the senses of sentences containing words for numbers. ([Dummett, 1993], p. 5)

We can therefore isolate the following:

Context Principle. The meaning of a word only has meaning in the context of a sentence.

So I can't just ask what the meaning of "3" is, I have to ask about the contexts in which "3" might legitimately appear.

We now ask:

Question. How can we analyse the meaning of words (in a particular context)?

One (very un-Wittgensteinian) idea (rejected by Dummett): Our knowledge of meaning is manifested by our ability to verbalise the meaning.

For example: I explain to you what I mean by the word 'dog': "By 'dog' I mean a typically quadrupedal mammal of the genus *canis* with such and such features."

But this sends us off on a regress. (What do I mean by 'mammal', 'four legged', '*canis*', 'genus' etc.?)

Instead, Dummett was greatly influenced by the following idea:

> For a large class of cases of the employment of the word 'meaning'—though not for all—this word can be explained in this way: the meaning of a word is its use in the language (Wittgenstein, *Philosophical Investigations*, §43)

Dummett followed Wittgenstein's lead here, arguing that:

> The meaning of a mathematical statement determines and is exhaustively determined by its use. ([Dummett, 1975], p. 6)

Let's call this the **Meaning as Use Principle** (I won't restate it, because Dummett's version is already pretty good!)

How should we understand this thesis? One response is:

Holism. The meaning of a term consists in its inter-theoretic connections with other terms, and permissions for use within them.

But he rejects this, to have this conception would require me to know the whole language, and that's clearly not how we operate.

Instead, for Dummett, to understand the meaning of a mathematical statement is just to *be able* to use the term correctly within communities of speakers in different contexts: How can we compute with numbers? How do we apply numbers to the real world and what kind of sentences do we assent to?

Given the Meaning as Use Principle, how should we understand truth? Dummett argues as follows:

> We are certain of the truth of a statement when we have conclusive grounds for it and are certain that the grounds which we have are valid grounds for it and are conclusive. ([Dummett, 1975], p. 6)

In mathematics, what's the way of showing a fact conclusively? Why proof of course!

> What we actually learn to do, when we learn some part of the language of mathematics, is

to recognise, for each statement, what counts as establishing that statement as true or as false. In the case of very simple statements, we learn some computation procedure which decides their truth or falsity: for more complex statements, we learn to recognise what is to be counted as a proof or a disproof of them...We must, therefore, replace the notion of truth, as the central notion of the theory of meaning for mathematical statements, by the notion of proof ([Dummett, 1975], p. 16)

For Dummett then:

1. An understanding of the meaning of a statement consists in the capacity to recognise a proof of it.

2. An understanding of anything smaller than a sentence (e.g. the term "the least natural number") consists in the computational ways it can contribute to a proof of a sentence.

Note that this immediately suggests the Brouwer-Heyting-Kolmogorov (BHK) interpretation we discussed last time:

A proof of $\phi \wedge \psi$ is a a pair of a proof of ϕ and a proof of ψ.

A proof of $\phi \vee \psi$ is a proof of ϕ or a proof of ψ.

A proof of $\phi \rightarrow \psi$ is a construction that turns a proof of ϕ into a proof of ψ.

A proof of $\neg\phi$ is a construction that turns the assumption of ϕ into a contradiction.

A proof of $\forall x\phi(x)$ is a construction that produces $\phi(a)$ for any object a.

A proof of $\exists x\phi(x)$ is the specification of an object a and a proof of $\phi(a)$.

This interpretation, as we discussed last time, validates intuitionistic logic; a sublogic of classical logic (every intuitionistic theorem is a classical theorem) on which the the Law of Excluded Middle and Double Negation Elimination can fail.

Question. How to understand axioms given that we're thinking of statements as supported by proof?

Well, axioms are just our starting point; they are what we accept as having one line proofs.

It's not clear whether or not we *had* to accept the axioms we do (and in this we might think Dummett, at least insofar as I'm reconstructing him, differs from the common understanding of axiom).

But maybe thinking creatures a lot like us are overwhelmingly likely to pick the axioms we do out of considerations of usefulness or similar.

3 Dissolving some problems

The Phenomenological Challenge is gone: Whilst there are difficult questions about use, it is a lot clearer than the idea of bare two-one-ness, pulling difficult questions about the ontology of mathematics down in to more tractable ones about language use.

The Identity Challenge is also avoided: The question of whether two mental constructions are the same comes down to questions of possible use, there's just no ontological question to answer.

Moe generally Dummett has very little time for these purely ontological questions:

> Realism is a metaphysical doctrine; but it stands or falls with the viability of a corresponding semantic theory...the context principle repudiates semantics. That principle ... ought therefore not to be invoked as underpinning realism, but as dismissing the issue as spurious. ([Dummett, 1991], p. 198)

For example even if the natural numbers exist "independently of the human mind" (and we can give meaning to that phrase!) we might still be intuitionists; the words we use mean the things they do insofar as we have capacities for use and proof.

So whether or not numbers are "out there" (or whatever) they are constructions insofar as our thought and language are concerned.

> Unlike material objects, mathematical objects are, on this thesis, creations of the human mind: they are objects of thought, not merely in the sense that they can be thought about, but in the sense that their being is to be thought of... ([Dummett, 1975], p. 19)

The Temporal Challenge is still open though.

This is a bit more difficult. Wasn't it *always* the case that there were (or weren't) two helium atoms circling the sun at some given distance 5 billion years ago?

> the appropriate generalisation of this [the account of truth via proof], for statements of an arbitrary kind, would be the replacement of the notion of truth, as the central notion of the theory of meaning, by that of verification; to know the meaning of a statement is, on such a view, to be capable of recognising whatever counts as verifying the statement. ([Dummett, 1975], p. 18)

We might then say the following:

Possible Verification. The truth of a past-tense sentence consists of its being the case that someone suitably placed could have verified it.[1]

With this in play, Dummett can say that there are two helium atoms because someone suitably placed could have verified (or falsified) that there are two such atoms.

Some interesting points here:

1. This allows Dummett to say that there's a large class of statements that the Platonist and Intuitionist can agree on; those that are *decidable*

[1]See here [Dummett, 2004], p. 44.

(could be verified by a proof in the relevant intuitionistic theory).

2. It perhaps opens him up to objections of the following form:

Isn't this 'ideal observer' highly mathematised?

Clearly they need to be seriously superhuman (check the number of atoms in a sphere five meters wide at the centre of the sun, for example).

But they can't be too superhuman...

Ixmucane decides she wants to create some numbers (say by observing intervals of space, or whatever). After half a second, she creates 0. Then after another quarter of a second, she creates 1. Then after another eighth of a second she creates 2,...and so on. After a full second, she's created infinitely many numbers (and can then observe an undecidable sentence in that structure).

The point is, in order to explain what an idealised observer looks like, don't we need to pack in mathematical limits (e.g. can't observe infinitely many things).

But this ideal observer is meant to be grounding our possible patterns of use for mathematical subject matter, which in turn explains the meaning!

We seem to need the resources we're trying to explain in doing the explaining.

4 Questions / Discussion

Question. Is there a non-mathematical view of idealised agents?

Question. Does it matter if the notion of idealised agent is highly mathematised?

Question. Is the idea of meaning as use too permissive in the mathematical context? (Is it just anything goes?)

Question. Is it possible to have my own private language of mathematics that can't be shared with anyone else?

Question. What do we *mean* by proof? Is an informal proof enough? Or should we expect a formal derivation?

Question. Are there links to other philosophical views here (e.g. Popper's falsificationism or verificationism more generally?)

Question. Is is possible to be both a Logicist and an Intuitionist?

References

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