Week 10: Intuitionism: Brouwer

Introduction to the Philosophy of Mathematics
Dr. Neil Barton
neil.barton@uni-konstanz.de

8 January 2020

Recap

Last time we saw that:

(1.) Cantor took the infinite very seriously as something that could be grasped by human minds (unlike many other thinkers, e.g. Locke, who thought that thought about the infinite was absurd).

(2.) He was one of the first to really target the infinite for mathematical analysis, defining the notion of an (infinite) cardinal number (two numbers have the same *cardinality* iff there is a bijection between them) and proved several facts about the infinite, including:

(i) The even numbers and the odd numbers (and indeed any infinite subset $A \subset \mathbb{N}$) has the same cardinality as $\mathbb{N}$.

(ii) $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality.

(iii) $\mathbb{N}$ is *strictly smaller* than $\mathbb{R}$.

(iv) (Cantor's Theorem) For any set X the set of all subsets of X ($\mathcal{P}(X)$) is *larger* than X.

(3.) The problem of proper classes (e.g. the Russell class $R = \{x | x \notin x\}$) remained for Cantor. His 'solution' was to argue that some collections/classes cannot be thought of "all together". He founded this distinction upon an idea of the *transfinite*, that which can be comprehended, and the *absolute infinite* (which for him had theological connotations and escaped characterisation).

This week, we'll look at how Cantor's views were received by the *Intuitionists*, and in particular Luitzen Egbertus Jan Brouwer (1881–1966), a Dutch mathematician and philosopher.

As we'll see, he had a very different attitude towards the infinite.

1 Platonism, Formalism, Intuitionism

Recall the platonist thesis that:

Platonism. Mathematical entities are (i) abstract, and (ii) mind-independent.

The paradoxes are very *troubling* for the Platonist.

Cantor was a platonist of sorts, he thought that talk about infinite sets referred to some real entities, in particular linked to theological considerations.

He gets himself into all sorts of trouble trying to make this difficult distinction between sets and proper classes (consistent and inconsistent multiplicities in his terminology).

One response (that we'll consider in detail next week):

Formalism. Mathematics is a meaningless game played with symbols.

e.g. I can play an arithmetic game by adding Hilbert strokes to one another (e.g. IIIII+II=IIIIIII), I can play a set-theoretic game by writing down some axioms and manipulating the symbols via derivation.

Formalism removes worries about what mathematical entities *are* by appealing to consistency; as long the formal axioms and rules I use are consistent, I can play any old mathematical game I want.

Brouwer felt this approach to mathematics did not respect its phenomenology.

Instead he proposed *Intuitionism*. We'll state it more precisely in a minute, but this is the idea that mathematics is somehow mind-dependent.

Kant writes the following in the *Proglomena*:

> Geometry bases itself on the pure intuition of space. Even arithmetic forms its concepts of numbers through successive addition of units in time.

So, for Kant, mathematics is *extracted* from sense experience. Brouwer takes up and develops this thought.

> The question where mathematical exactness does exist, is answered differently by the two sides; the intuitionist says: in the human intellect, the formalist says: on paper. ([Brouwer, 1913], p. 83)

2 The epistemology and metaphysics of Intuitionism

Brouwer thought that Kant was wrong about geometry: There are non-Euclidean geometries, and so many different ways of representing our experience of space.

He did think Kant was correct about arithmetic, however:

> This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts,..., the intuition of the bare two-oneness. This intuition of two-one-ness, the basal intuition of mathematics, creates not only the numbers one and two, but

> also all finite ordinal numbers... ([Brouwer, 1913], p.)

So, for Brouwer, our knowledge of arithmetic is obtained by reflecting upon the way that our experience is phenomenologically structured through time.

He also felt that we were able to get an idea of continuum from our phenomenology:

> Finally this basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i. e., of the "between," which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units. ([Brouwer, 1913], p. 86)

So for Brouwer, our idea of the continuum flows from the idea that we can always interpose a point between any two others.

Remark. These are very Aristotelian ideas!

Brouwer held that the objects of mathematics are *mental constructions* derived from our *intuition*.

In this sense, we cannot say that a mathematical object exists until we have actually constructed such an object.

Such a construction will take the form of a proof (possibly non-formal!—bear in mind that mathematics is a process by which we intuit objects for Brouwer).

To bring everything together, we can define:

Brouwerian Intuitionism holds that:

(i) Mathematical reality comes into being with our acts of *mental construction*.

(ii) There is no distinction between what is true in mathematics and what has been proved.

(iii) There is no untensed truth.

3 Intuitionistic logic and mathematics

What kind of logic and mathematics does Brouwer's intuitionism support?

Brouwer came up with his own version of logic: *intuitionistic logic*.

Key here is that we interpret the connectives as algorithmic claims about proofs (the so called Brouwer–Heyting–Kolmogorov interpretation).

A proof of $\phi \wedge \psi$ is a a pair of a proof of ϕ and a proof of ψ.

A proof of $\phi \vee \psi$ is a proof of ϕ or a proof of ψ.

A proof of $\phi \rightarrow \psi$ is a construction that turns a proof of ϕ into a proof of ψ.

A proof of $\neg\phi$ is a construction that turns the assumption of ϕ into a contradiction.

A proof of $\forall x\phi(x)$ is a construction that produces $\phi(a)$ for any object a.

A proof of $\exists x\phi(x)$ is the specification of an object a and a proof of $\phi(a)$.

Some features:

The Law of Excluded Middle Fails. The Law of Excluded Middle states that for any formula ϕ, $\phi \vee \neg\phi$ is a logical law. This fails in intuitionistic logic, since in order to use $\phi \vee \neg\phi$ I have to have *actually* proved one of ϕ or $\neg\phi$. But this isn't always the case in classical logic. Consider the following proof (given by Linnebo):

Theorem. (Classical logic) There are irrational numbers a and b such that a^b is rational.

Proof. Consider $\sqrt{2}^{\sqrt{2}}$. Is it rational? If yes, then $a = b = \sqrt{2}$ does the job. If no, then $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ do the job since:

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}.\sqrt{2})} = \sqrt{2}^{2} = 2$$

(by tedious algebraic manipulation)

But we haven't *actually* constructed such a pair of a and b here (I didn't show you which objects could be constructed that would do the job).

Double negation elimination fails. Double negation elimination is the law $\neg\neg\phi \rightarrow \phi$. But this fails in intuitionistic logic, since if I show you that from the assumption of $\neg\phi$ I can show a contradiction (i.e. $\neg\phi$ isn't provable) I haven't thereby exhibited a proof of ϕ. Consider the following example:[1]

Prosecutor: "The defence says I can't prove the defendant is guilty. But they can't prove that."

Defence: "This is true."

Prosecutor: "Therefore he is guilty."

Jury: "Agree."

Optional Exercise. Consider two common forms of proof by contradiction:

Reductio ad absurdum states that if I assume ϕ, and I prove a contradiction, I can infer $\neg\phi$.

The Principe of Indirect Proof (PIP) states that if I assume $\neg\phi$ and prove a contradiction, then I can conclude ϕ.

Question. Which one of those two does the Intuitionist reject? Why?

That's some features of the logic (underlying rules of inference) for the intuitionist.[2]

Arithmetic is largely the same, but with intuitionistic logic instead of classical and a conception of the numbers as forever in a process of construction.

Heyting Arithmetic has the same non-logical axioms as **PA**, but the under-

[1]This example is taken from the discussion here: https://math.stackexchange.com/questions/2169303/natural-language-examples-for-failure-of-double-r

[2]Careful with terminology! Some authors refer to this as reductio too. Precisely because intuitionists differ on the matter, I prefer to call them both proofs by contradiction but distinguish reductio and PIP.

lying logic is intuitionistic.

This has implications for how we show generalised theorems. If I want to show $\forall x \phi(n)$, it's not enough to show that if I assume $\neg \forall x \phi(n)$ I get a contradiction, I need an actual method that will prove $\phi(n)$ for any particular n you give me.

Brouwer proposes a *complete rejection* of uncountable infinities. This is how he avoids the paradoxes, talk of large infinite sets and paradoxical collections is completely meaningless.

There is then the question of how we deal with the *real numbers* (since these are often understood as *infinite* sequences).

Brouwer's response: We can think of these as *choice* sequences, they are not actually infinite, but rather ways of specifying continuations of finite sequences.

That's easy enough when we have a rule (what are called *lawlike* sequences), e.g. $0, 2, 4, 6, 8, ..., 2n, ...$

But what about seemingly *random* sequences e.g. $9, 2, 3, 5, 6, 8, ...$.

These can be thought of as *free* choice sequences: Think of a never-ending sequences of coin-flips (or if you want base 10 notation, rolling a d10 die).

So Brouwerian Intutionism can be provided with its own interesting logic and mathematics.

4 Challenges for the view

Let's start with one strength of the view:

Benacerraf's Epistemological Challenge. [Benacerraf, 1973] Given that numbers are non-spatio-temporal, acausal entities, how do we gain knowledge of them?

Response. Numbers aren't non-spatio-temporal or acausal, we *literally* construct them, and they are just *part* of our experience.

Now let's move on to some more significant problems:

Provability Challenge. Surely there are unproved but provable propositions?

Response. What do you mean by "ϕ is provable"?

If you mean "there is a proof of ϕ" then there's no unproved but provable mathematical propositions.

If you mean "the possibility of proving ϕ has not yet been refuted" the fact that an unproved sentence is provable does not entail that there *exists* a proof awaiting discovery. (Remember intuitionists *mean* something very different with their logic from the usual material interpretation! Things have to be *constructed*.)

Revisionary Challenge. Intuitionism is strongly revisionist about mathematical practice.

Depending on the kind of programme you want, this would have been utterly untroubling for Brouwer (in fact he quite liked this). He was aiming to revise mathematics, and intuitionistic mathematics can be provided with corresponding formal theories (for e.g. number theory, analysis, and set theory).

Phenomenological Challenge. This talk of bare-two-one-ness is at best perplexing, and at worst incoherent (especially when we want a foundations for *mathematics*, which is meant to be especially secure).

Identity Challenge. Let's grant some kind of interpretation of bare-two-one-ness and mental construction. How many 2s are there? Is my 2 the same as your 2? What happens if we construct *the same* proof? How do we account for that claim?

Temporal Challenge. Wasn't it *always* the case (or not) that there were (or weren't) exactly two helium atoms orbiting the sun? What does any *actual* construction have to do with it?

(**Possible response.**) We're considering what *could* have been constructed. But this notion of ideal construction looks like it would need mathematical analysis, and this seems problematic from an intuitionistic perspective (more on this next week).

It thus seems that Brouwerian Intuitionism, whilst it has some strengths, is pretty problematic (philosophically speaking).

Let's be clear though: While Brouwer's intuitionism faces cogent philosophical objections, the fruit it bore was impressive; it turns out (via something called the Curry-Howard correspondence) that intuitionistic logic is linked to kinds of type theory, and this has applications in functional programming.

> what would happen to an employee who, in response to a request that he write software to perform a certain computation, presented his boss with two programs and the information that, although one of those programs performed the required computation, nobody could ever tell which one? (Preface to [Bridges, 1994])

5 Questions

Great questions again this week. I've tried to include as many as I can

Question. What does Brouwer mean by "two-one-ness"? Is that *really* how our arithmetical concepts are grounded?

Question. Does it matter if Brouwer's epistemological story is *literally* true?

Question. Does Intuitionism undermine the certainty of mathematics?

Question. Does Intuitionism just make mathematics part of psychology?

Question. Is Intuitionism incompatible with Formalism?

Optional Exercise. What are the axioms we have discussed in the course that the Intuitionist might object to?

References

[Benacerraf, 1973] Benacerraf, P. (1973). Mathematical truth. *The Journal of Philosophy*, 70(19):661–679.

[Bridges, 1994] Bridges, D. S. (1994). *Computability: A Mathematical Sketchbook*. Springer.

[Brouwer, 1913] Brouwer, L. E. J. (1913). Intuitionism and formalism. *Bulletin of the American Mathematical Society*, 20:81–96.