Week 9: The Cantorian Infinite

Introduction to the Philosophy of Mathematics
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Recap

Last time we saw that:

1. Frege's system for founding mathematics as part of logic turned out to be inconsistent.
2. Russell and Whitehead's system of Simple Type Theory seemed too restrictive and faced general philosophical problems.
3. The Neo-Logicist position that Hume's Principle is analytic and that this yields a slightly restricted version of Logicism is a live (though controversial) position.

This week we'll examine the seminal work of Georg Cantor (1845–1918) concerning the infinite and his attitudes to the paradoxes.

1 Worries about the infinite

We should first note that the infinite was *not* well regarded in the period leading up to Cantor's discoveries.

Recall that Aristotle's view was that it only made sense to speak of the infinite as potential rather than actual (in addition or division).

This view had gained something of an orthodoxy by the time Cantor was writing.

Mathematicians studied *finite* objects (numbers, lines, etc.) that are only infinite insofar as they can be indefinitely extended.

But the *objects* of mathematics should not be 'indefinite' in any sense.

Locke on the subject sums up the prevailing sentiment well:

> (If) a man had a positive idea of infinite..., he could add two infinities together: nay, make one infinite infinitely bigger than another, absurdities too gross to be confuted.[a]
>
> [a] *An Essay Concerning Human Understanding* (Volume 1), Book II, Chapter XVII (p. 292).

As a species of this thought, let's consider Galileo's Paradox (a similar version was discovered independently by Thābit ibn Qurra al-Ḥarrānī).

Galileo's Paradox. Consider the following two sets:

$$\mathbb{N} = \{n | \text{“}n \text{ is a natural number”}\}$$

$$Sq(\mathbb{N}) = \{n^2 | n \in \mathbb{N}\}$$

The 'paradox' arises from considering which set is larger? 'Clearly' $\mathbb{N}$ is larger, since every member of $Sq(\mathbb{N})$ is a member of $\mathbb{N}$, but not vice versa. But 'clearly' they have the same size, since we can pair up $Sq(\mathbb{N})$ with $\mathbb{N}$ one-to-one using the function $f(n^2) = n$.

Cantor's insight was to treat infinities seriously, and carefully disambiguate two senses of 'size' at play in Galileo's paradox.

In doing so he developed the modern theory of infinite *cardinals*.

2 Cantor's insight

Cantor's original motivation wasn't Galileo's paradox, however.

Rather he was motivated by *mathematical* concerns; he wanted to represent *functions* using particular kinds of *series*.[1]

We first need the following idea:

Definition. Given two classes A and B, a total function $f : A \to B$ is a *bijection* iff:

(i) f is *surjective* or *onto* (that is $(\forall b \in B)(\exists a \in A) f(a) = b$, i.e. f 'covers' B with members of A).

(ii) f is *injective* or *one-to-one* (that is $(\forall a_0, a_1 \in A)[f(a_0) = f(a_1) \to a_0 = a_1]$, i.e. f doesn't map two distinct objects in A to the same object in B).

Informally: Bijections establish *pairing offs* of sets with one another.

Some examples of bijections: the identity function, the function that maps the corresponding fingers on my right and left hands, the natural map from $\mathbb{N}$ to the even numbers (given by $f(n) = 2.n$).

Definition. Two sets A and B have the same *cardinality* or *Cantorian size* iff there is a bijection between them.

Note that there are lots of infinite sets with the same Cantorian size; the even numbers, the odd numbers, the natural numbers, the squares.

Solution to Galileo's paradox: There are (at least) two notions of 'size'; being a proper subset and having the same Cantorian size. These are just different notions, and we shouldn't think there's any contradiction in two sets having the same Cantorian size but one being a proper subset of another.

[1] A *trigonometric series* is of the form $\frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$. Cantor showed the uniqueness of representations of certain functions by trigonometric series in 1869.

In fact, a set having the same cardinality as one of its proper subsets can be used to give a definition of infinity:

Definition. A set A is *Dedekind infinite* iff there is a proper subset $A' \subset A$, such that there is a bijection $f : A' \to A$.

Galileo's paradox thus depends on conflating two notions of size: The Cantorian notion of pairing off their elements, and the 'Euclidean' idea that the whole should always be greater than the part.

A few amazing results concerning cardinalities:

Theorem. The rational numbers $\mathbb{Q} = \{\frac{m}{n} | m, n \in \mathbb{N}\}$ and the natural numbers $\mathbb{N}$ have the same cardinality.

Remark. This is despite the fact that $\mathbb{Q}$ is *dense*: Between any two rational numbers there is a third (and hence between any two rationals there are infinitely many rationals).

Theorem. There are as many real numbers in the interval $(0, 1)$ (or indeed between any two real numbers) as there are in the whole real line.

This latter fact (actually the fact that there are as many points on one side of a square as in the whole square) led Cantor to write (in a letter to Richard Dedekind):

> Je le vois, mais je ne crois pas. (I see it, but I don't believe it.)

These results might lead us to question whether or not all infinities are the same size. The prevailing opinion is no, as shown by:

Theorem. There are more real numbers between 0 and 1 than there are natural numbers.

In fact we have more generally:

Theorem. (Cantor's Theorem) Let X be a set. Define the *powerset of* X (or just $\mathcal{P}(X)$) to be the set of all subsets of X (i.e. $\{Y | Y \subseteq X\}$). Cantor's Theorem states that $\mathcal{P}(X)$ always has greater cardinality than X (i.e. there is no bijection between X and $\mathcal{P}(X)$).

Cantor's Theorem is thus very interesting: It shows that assuming we can always take powersets there is a whole hierarchy of ever bigger and bigger infinities.

These infinities are often referred to as indexed by the $\aleph$ (aleph) function: So $\aleph_0$ is the smallest infinity, then $\aleph_1$, $\aleph_2$, ..., etc.

Cantor in fact introduced a whole calculus for these infinite cardinals; they can be added, multiplied, and exponents taken.

Cardinality considerations can be useful in proving mathematical and logical theorems, e.g.:

Optional Exercise. We say that a real number is *transcendental* iff it is not a solution of a nonzero polynomial equation with integer coefficients. Show that Cantor's Theorem implies that there are transcendental reals.

Hint. How *many* nonzero polynomial equations with integer coefficients are there?

3 Cantor and the Paradoxes

It was essential to Cantor's outlook to treat infinities as actual and legitimate objects of mathematical inquiry:

> Thus, each potential infinite, if it is rigorously applicable mathematically, presupposes an actual infinite.

But if we are to treat infinities as objects of mathematical study, it is imperative that we answer the question of paradoxical collections.

Cantor was indeed very troubled by the paradoxes, in fact he gives his name to one:

Cantor's Paradox. There is no universal set $\{x|x = x\}$.[2]

Why is this so? Well if there were such a set U we could consider its powerset $\mathcal{P}(U)$...

Cantor's solution was to make a sharp distinction between *consistent* and *inconsistent* multiplicities (classes):

> For a multiplicity can be such that the assumption that all of its elements "are together" leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as "one finished thing". Such multiplicities I call absolutely infinite or inconsistent multiplicities... If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as "being together", so that they can be gathered together into "one thing", I call it a consistent multiplicity or a "set". ([Cantor, 1899], p. 114)

Cantor's solution is thus to say that there are (at least) two different kinds of collection-like objects: *Sets* and *Proper Classes*.

But what exactly is the difference between them meant to be?

The proper classes are those collections that, if we assume they are sets, we get a contradiction.

But is this enough?

Or do we need an account of *why* 'inconsistent multiplicities' are contradictory?

In many ways, proper classes look very much like sets, they are extensional (presumably) in that two proper classes are identical if they contain the same elements.

So what stops them from being sets?

Cantor's answer was theological in character.

He argued that the Absolute (parts of which are the inconsistent multiplicities) are manifestations of God and unknowable, whereas the transfinite is knowable by us.

[2] Similar versions arise from there being no largest infinite size, or considering the class of all $\aleph$-numbers.

> It belongs particularly to speculative theology to investigate the absolute infinite and to determine what can be humanly said about it. On the other side, the questions about the transfinite belong chiefly to the domain of metaphysics and mathematics. (Cantor 1887-88, p. 378)

We might be worried about the apparent theological dependence here. But perhaps we can put things in more neutral terms: The absolute is that which evades human understanding.

> The Absolute can only be acknowledged and admitted, never known, not even approximately. ([Cantor, 1883], p. 205)

This seems problematic though. Prima facie, there are meaningful and true claims I can make about proper classes.

Example. Any proper class is a subclass of the universal class.

This seems to conflict with Cantor's idea that the absolute cannot be known.

A different interpretation would be:

Generality Relativism. We cannot talk about *all* sets, rather we only ever talk about some fixed set of sets.

The 'universal set' of one domain is then just a regular set in a larger domain.

Possible problem. Denying the intelligibility of quantification over *all* objects of a certain kind seems to be self-undermining.[3]

Dealing with the problem of proper classes is a very difficult live philosophical and mathematical problem.

After the break, we'll see two different attitudes to the transfinite; the revisionary accounts of the *intuitionists* and Hilbert's *formalism*.

[3]A lot of literature focusses on how to get around this problem. See e.g. [Linnebo, 2010].

4 Questions/Discussion

Question. Is Cantorian size a good way of capturing the informal notion of 'size'?

Question. Is Cantor's distinction between consistent and inconsistent multiplicities satisfactory?

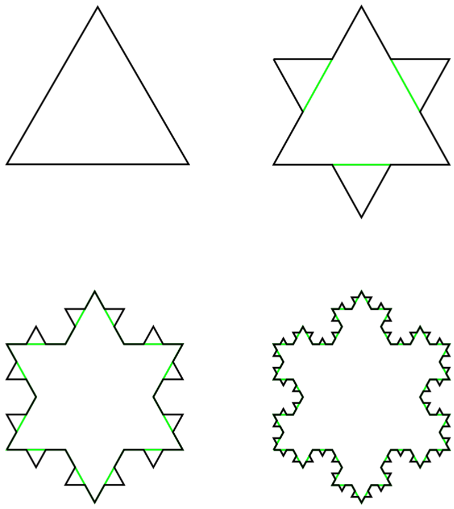
Question. Does every potential infinity presuppose an actual infinity? Was Cantor in some sense a potentialist about infinity (i.e. did he think that some collections could not be completed)?

Question. Given their different sizes, is there a substantial difference between $\mathbb{R}$ and $\mathbb{N}$?

Question. Note that because $\mathbb{N}$ is *countable* I could in principle *define* any element of $\mathbb{N}$. I letting s stand for "the successor of" I have to write s n-many times before 0 to define any element of $\mathbb{N}$ (obviously this is practically not possible). Given a countable language, however, there are elements of $\mathbb{R}$ that I cannot define in that language. Is this important for us?

Question / Optional Exercise. *Hilbert's Hotel.* Suppose you have a hotel with $\aleph_0$-many single rooms. They are all full. Suppose someone else turns up. How can you accommodate them in the hotel? (You may ask people to move rooms.) Suppose a further $\aleph_0$ many people turn up the following day. How can you accommodate them? What if $\aleph_1$-many people turn up?

Season's Greetings![4]



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[van Heijenoort, 1967] van Heijenoort, J., editor (1967). *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*. Harvard University Press.

[4] Koch Snowflake credit: CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=1898291