Week 7: Neo-Logicism

Introduction to the Philosophy of Mathematics
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Recap

Last week we saw:

1. Frege's Logicism ran into the difficult problem of Russell's paradox.

2. Russell's attempted solution was to argue that logic should respect *typing* into individuals (type 0), classes of individuals (type 1), classes of classes of indivuals (type 2), and so on.

3. However, the Simple Theory of Types is problematic for several reasons (e.g. meaningful conditions apply to more than one type, the doctrine looks self-undermining, we seem to want to make cross-type comparisons in comparing cardinality).

This week, we'll examine some contemporary attempts to recover some intuitions from Frege and Russell.

The idea will be the following: While full-blown Logicism might be too strong, the idea that arithmetical truths are *analytic* and can be obtained from *abstraction principles* is still on the right track.

1 Neo-Logicism: The Basic Idea

Let's walk back a little.

Recall that Frege thought that numbers were *abstract* (i.e. not spatial or temporal):

> ...but where is the number 4? It is neither outside us nor within us...Not every object has a place. (Frege, *Grundlagen*, §61).

...and that he is sensitive to the following observation that has popped up constantly in our course:

> "How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them?" (Frege, *Grundlagen*, §62)

In trying to solve this, he argued that numbers were part of logic, and derived the basic principles of arithmetic from a version of second-order logic augmented with an axiom about how concept extensions behaved called 'Basic Law V'.

Our access to numbers for Frege is thus given by logic.

Unfortunately his system was inconsistent.

Russell and Whitehead's Simple Theory of Types, while mathematically and practically useful, ran into several philosophical and technical problems in executing the Logicist thesis properly.

Neo-Logicists give up the idea that we should derive mathematics from logic *simpliciter*.

Rather we should view mathematics in the following terms:

1. We *should* think of numbers as *logical objects*.

2. They are *characterised* by *abstraction principles*.
3. Some of these abstraction principles are *analytic*; i.e. true in virtue of the meaning of the concepts involved.
4. Our *knowledge* of numbers is thus given by *logic*; they are sui generis objects and there are analytical truths about them.

Frege was thus right (according to the Neo-Logicist) that the objects of arithmetic (and quite possibly other areas) were logical and could be understood through logic, we just have to be careful what we mean by this.

We'll focus on two great Neo-Logicist warriors: Bob Hale (1945-2017) and Crispin Wright (b. 1942).

2 Abstraction Principles

First, we need to examine *abstraction principles*.

It is not clear that there is a unique kind of principle denoted by the term "abstraction principle" (it may be more of a family resemblance concept[1]).

We therefore just need to be a touch careful when discussing this notion, considering principles on their own merits, rather than getting too hung up on whether this or that principle deserves the term 'abstraction principle'.

Recall that *abstraction* concerns the idea that we ignore certain specific properties of objects in order to get at something more general.

e.g. I can talk about a line in a space, but forgetting its specific position and length etc., I can talk about its *direction*.

An *abstraction principle* then comes up with criteria of identity between objects on the basis of some similarity between them.

[1]See, for example, the discussion of the notion *game* in §66 and §67 of Wittgenstein's *Philosophical Investigations*.

These kinds of similarities are understood via *equivalence relations*.

These are binary relations that are:

(i) **Symmetric:** if bRa then aRb.

(ii) **Reflexive:** aRa for any a.

(iii) **Transitive:** If aRb and bRc, then aRc.

Some examples: identity, sameness of height, sameness of cardinality.

An *abstraction principle* can be viewed as a way of linking properties of objects under some equivalence relation to more abstract entities. For example:

Abstraction for Lines. (In Euclidean geometry) $d(l_1) = d(l_2) \leftrightarrow l_1 \| l_2$ (The direction of l_1 is equal to the direction of l_2 if and only if they are parallel.)

The Neo-Logicist wants to find an abstraction principle for arithmetic.

But they already have one in Hume's Principle!

Definition. *Hume's Principle* (HP) is the following statement (given an operation $\sharp$ that maps concepts to objects):

$$\sharp x F(x) = \sharp x G(x) \leftrightarrow F \approx G$$

i.e. The number of Fs is the same as the number of Gs iff there is a bijection between the Fs and the Gs.

They then claim that Hume's Principle is *analytic* of the concept *cardinal number*; it is *logically true* and *analytic* that the numbers obey Hume's Principle.

If this is the case, we can use Frege's Theorem:

Theorem. (Frege's Theorem) The usual axioms of second-order logic plus HP suffice to derive PA_2.

...thereby arguing that given our concept of number, we can know the theorems of arithmetic (well, $\mathbf{PA}_2$ at least) on the basis of logic and our concept of number.

> And if in addition the Principle may be viewed as a complete explanation—as showing how the concept of cardinal number may be fully understood on a purely logical basis—then arithmetic will have been shown up by Frege's Theorem not as part of logic, it is true, but as transcending logic only to the extent that it makes use of a logical abstraction principle—one whose right-hand side deploys only logical notions. Crispin Wright, *Frege's Conception of Numbers As Objects* (1999) p. 279–280

3 Assessment

There's a lot to be said about Neo-Logicism, and work is ongoing (some of it gets quite technical).

Good Point 1. We can still do the bootstrapping trick (sometimes referred to as Frege's Trick), since Hume's Principle (as interpreted by the Neo-Logicist) introduces numbers as new entities.

The operation $\sharp$ maps concepts to *first-order* objects of the domain.

So we can still get 0 by considering the concept defined by $x \neq x$.

We get 1 by considering the concept $x = 0$.

We get 2 by considering the concept $(x = 0 \vee x = 1)$

We get 3 by considering the concept $(x = 0 \vee x = 1 \vee x = 2)$

...and so on...

The difference with Frege's version is that he could *define* the number n as $\{\{x|F(x)\}|$"F has n-many instances" $\}$

Here we just assume that they exist and are governed by HP.

Remember: The bootstrapping argument was blocked under the Simple Type Theory approach, and we had to use an axiom of infinitely many individuals to get it to work.

Moreover, the typing resulting in the fact that we had many 0s 1s, ..., ns,...

Good Point 2. Since HP is true on some infinite domains, the technical theory given is consistent iff $\mathbf{PA}_2$ is.

Some problems for the view:

Problem 1. *The Julius Caesar Problem.* What is the truth value of $\sharp x F(x) =$ Julius Caesar (or some other random physical object)?

Hume's Principle appears to have nothing to say here, despite the intuitive falsity of the claim.

Response. (Hale and Wright) When we learn that an identity criterion applies to some object x but not another y, we learn that x has a property that y lacks, and hence (by the Indiscernibility of Identicals) $x \neq y$.

Problem 2. Is Hume's Principle really analytic and/or logical?

Some points against:

Sub-Problem 2.1. Hume's Principle is only true on infinite domains, but analytic truths are those which hold on any domain.

Response. (Wright) Analytic truths are what hold given logic *and* definitions.

There are infinitely many objects *given* the definition of Hume's Principle as the status of a definition.

This is just the Logicist contention; so to deny it amounts to question-begging.

Sub-Problem 2.2. Hume's Principle has some *staggering* commitments.

It's purview is not just arithmetic, but cardinals more widely.

For example, 0 is still defined via $\sharp x(x \neq x)$.

But what then about *the universal number* or *anti-zero*: $\sharp x(x = x)$?

All things, taken together, have a number!

This is a very strong claim, and perhaps counts against its analyticity.

Two options:

1. Bite the bullet. It's analytic that there is a universal number. The hard work is then making that palatable.

2. Argue that not all predicates define the right kind of concept for being numbered. You then need either a different account of zero (or explain why the one you have is still OK), and a principled account of what differentiates the good from the bad.

Debate is ongoing, and there's a lot of new work of both a philsophical and technical character (e.g. so called 'Bad Company' and 'Good Company' objections that we haven't gone in to today, but can discuss in questions if you like).

Neo-Logicist ideas have also been extended to parts of analysis and set theory.

It's thus a live position, though unclear exactly how it should be filled out.

4 Questions/Discussion

Question. Is Hume's Principle actually analytic?

Question. More generally, are abstraction principles just ways of *recarving content* (as the Neo-Fregeans would say)?

Question. One purported solution for avoiding the universal number is to say that we never quantify over all objects. There is a universal number for every domain, but not for all domains. What do we think of this view?

Question. One way of viewing the above problem is to think of Hume's Principle as *dynamic* and *modal* (you always introduce new objects). What do we think of the idea of modality in mathematics?