Week 6: Logicism: Russell and Whitehead

Introduction to the Philosophy of Mathematics
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Recap

Last week we saw:

1. Frege tried to argue that arithmetic was part of logic.
2. In doing so, he embarked upon the ambitious programme of deriving the axioms of $\mathbf{PA}_2$ from his second-order logic.
3. Unfortunately his system contained a contradiction.
4. He tried to rescue it by weakening Basic Law V. His solution was ad hoc (he was aware of this, the decision was made under a publication deadline) and was inconsistent anyway.

This week we'll see an important attempt to rescue Frege from contradiction given by Betrand Russell (1872–1970) and Alfred North Whitehead (1861–1947).

It's especially interesting because they[1] mixed *philosophical* considerations with *technical* ones.

Also, we start to see a theory (Type Theory) that has enormous implications for how mathematics developed.

1 The Shape of a Solution

Certainly Russell held a very strong logicism:

> The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age; and when this fact has been established, the remainder of the principles of mathematics consists in the analysis of Symbolic Logic itself.[a]
>
> [a]This is from a book Russell wrote called *The Principles of Mathematics* (p. 5) and is <u>not</u> the same as *Principia Mathematica*.

He was also definitely troubled by his eponymous paradox, as he writes in autobiography:

> I was trying hard to solve the contradictions mentioned above. Every morning I would sit down before a blank sheet of paper. Throughout the day, with a brief interval for lunch, I would stare at the blank sheet. Often when evening came it was still blank. (From Russell's autobiography, p. 142)

Why is Russell's Paradox so troubling for the Logicist?

It looks like some sort of class theory is a natural choice for underwriting logicism. (At least, that was Frege and Russell's position.)

If classes are somehow 'special' objects, then the theory of them will lack the topic-neutrality associated with logic, and thus mathematics will not be part of logic.

Therefore, in order to motivate the claim that the theory of extensions (and hence mathematics) is part of logic, the Logicist needs to

[1]Especially Russell, and we'll focus on his attitudes since he was a bit more philosophically active.

motivate logical principles governing classes that will have to block the paradoxes.

But this looks difficult, since in order for the theory of classes to be part of logic, it looks like we need:

1. Every meaningful concept should have an extension (without further argument, this includes a concept corresponding to non-self-membership).
2. Extensions should be governed by the rule $y \in \{x|F(x)\}$ iff $F(y)$.

But this amounts to the Comprehension Principle required for Russell's paradox!

Russell's 'solution' was to argue that some of the predicates in Frege's language (in particular $x \notin x$) are not meaningful, and have no place in a logically perfect language.

But it's one thing to say this, and another to motivate it. Russell did so through what he called *logical types*.

Basic Strategy:

1. Motivate the claim that a logically perfect language would have to respect the idea that reality is stratified into distinct *logical types* (which we'll talk about below).
2. Use this notion of typing to come up with a theory which (i) respects typing, (ii) prevents the paradoxes, and (iii) in which we can develop the whole of mathematics.

2 Logical Types: Blocking the paradoxes

Roughly speaking the distinction between types is whether or not objects are individuals, classes of individuals, classes of classes of individuals, etc...

First Motivation. Mathematicians routinely identify different types of objects in their reasoning.

e.g. Let G be a group and a be an element of G. Now consider some b...

Second Motivation. We are sensitive to errors of logical type. Consider the following sentences:

(i) "П" is a member of the Cyrillic alphabet.

(ii) The Cyrillic alphabet is a member of the Glagolithic alphabet.

(iii) The Cyrillic alphabet is a member of the *class* of Slavonic alphabets.

(ii) fails to respect type distinctions: **Why?**

The argument is then that things fall into *logical types* where:

Type 0: Individuals (e.g. Fido, Kepler-62e, П)

Type 1: Classes of individuals (e.g. *C.I. familiaris*, solar systems, the Cyrillic alphabet)

Type 2: Classes of classes of individuals (e.g. *C. Lupus*, galaxies, the class of Slavonic alphabets)

Type 3: Classes of classes of classes of individuals (e.g. *Canis*, universes of astrophysics, the class of all alphabet families)

...and so on...

We now say:

Typing of Objects. Every object belongs to exactly one logical type.[2]

Typing of Grammar. Every singular term has a syntactical type, namely the type of the object it denotes. Every variable has a single type; the type of the objects over which it ranges.

Objectual Type Membership. An object x can only be a member of an object y iff the type of y is *exactly one* above the type of x.

Typing of Syntactic Membership Claims. We can then say that for singular terms α and

[2]Also there are no infinite types, but we set this aside for now.

β, the syntactic type of β must be *exactly* one above that of α for $\alpha \in \beta$ to be well-formed.

More generally:

Doctrine of Type Predication. A 1-place predicate is true or false of only entities of a single type. So every 1-place predicate $F(x_k)$ is true of only entities of type k.

Returning to our earlier example: The Cyrillic alphabet (type 1) is a member of the Glagolithic alphabet (type 1) fails to express a proposition: It does not respect typing considerations.

Recall the reasoning of Russell's Paradox:

1. We consider the Russell Class $R = \{x|x \notin x\}$.
2. By Comprehension we have $y \in \{x|x \notin x\} \leftrightarrow y \notin y$.
3. We now substitute in R for y and obtain $\{x \mid x \notin x\} \in \{x \mid x \notin x\} \leftrightarrow \{x \mid x \notin x\} \notin \{ x \mid x \notin x\}$. Contradiction!

Block 1. The condition $x \in x$ is not meaningful (by typing), and so nor is $x \notin x$, neither correspond to a legitimate concept, and so $x \notin x$ cannot be used in the Comprehension Principle (Line 2).

Block 2. Moving from line 2 to line 3 involves a substitution of the following form. We want to substitute $\{x|F(x)\}$ for y in a sentence with the form $F(y) \leftrightarrow y \in \{x|F(x)\}$, to obtain a sentence with the form $F(\{x|F(x)\}) \leftrightarrow \{x|F(x)\} \in \{x|F(x)\}$. But this also violates typing.

3 Logical Types: Saving Logicism?

Recall that for Logicism to be maintained, Russell needed to maintain the Comprehension Principle for mathematics to continue to be part of logic.

We *do* have a Comprehension Principle for every type:

Typed Comprehension. For any predicate $F(x_k)$ and any entity w_k:

$$w_k \in \{x_k|F(x_k)\} \leftrightarrow F(w_k)$$

We just can't use these comprehension principles to generate the contradiction.

Using these axioms (plus some other ones we'll see below) Russell and Whitehead were able to reconstruct, in painstaking detail, the theorems of arithmetic in 'logic'.

The level of rigour is impressive (one might say extreme). They prove the theorem that $1 + 1 = 2$ on p. 86 of the second volume of *Principia Mathematica*, accompanied by the following comment:

> "The above proposition is occasionally useful"

4 Problems

There are several problems with the doctrine of types:

Problem 1. There are intelligible conditions that apply to entities of more than one type. e.g. "x is a finite class".

Problem 2. There are classes that plausibly contain entities of more than one type (e.g. *C. Lupus*).

Problem 3. *Gödel's Objection.* The doctrine is self-undermining. (e.g. "Every object belongs to exactly one logical type.")

Let's distinguish two claims:

(1.) Does the doctrine of types provide a satisfactory solution of Russell's Paradox?

(2.) Can an axiomatic theory that respects type grammar be used to establish the thesis that mathematics is logic?

Thus far we've only argued against (1.).

However, there are reasons to be suspicious of (2.) as well.

Problem. *The Need for Non-Logical Axioms.* Recall how Frege got infinitely many natural numbers:

Bootstrapping. Frege defined the natural numbers as:

$0 =_{df} \{\{x|F(x)\}|$"F has no instances" $\}$

$1 =_{df} \{\{x|F(x)\}|$"F is equinumerous with $\{0\}\}$.

$2 =_{df} \{\{x|F(x)\}|$"F is equinumerous with $\{0, 1\}$" $\}$.

$3 =_{df} \{\{x|F(x)\}|$"F is equinumerous with $\{0, 1, 2\}$"$\}$.

and more generally: $n =_{df} \{\{x|F(x)\}|$"F has n-many instances" $\}$

We then get the existence of an infinite set for free by taking the predicate "$\mathbb{N}(x)$" standing for "x is a natural number" (this can be defined in Frege's system) and considering:

$$\{x|\mathbb{N}(x)\}$$

But all these look like classes of mixed type!

To get around this, Russell assumed the following axiom:

Axiom of Infinity for Individuals. There are infinitely many individuals.

But is this a principle of *logic*? (Recall the Logicist belief that logic is meant to be topic neutral—now we're saying that *every* legitimate domain is infinite!)

There are other axioms that are *mostly* accepted in mathematics that were not clearly logical.

The Multiplicative Axiom. (or as we call it these days: *The Axiom of Choice*) For any class K of pairwise disjoint non-empty classes, there is a class C that shares exactly one member with each element of K.[3]

[3] The Axiom of Choice, despite being regularly used in a huge amount of mathematics, has attracted a lot of controversy. See for a humorous take: `https://xkcd.com/982/`.

The status of these axioms as principles of *logic* might be disputed.

Problem. Even with the Axiom of Infinity for Individuals, there are many empty sets, and many representatives for our mathematical objects throughout the types:

> ... 0 and 1 and all the other cardinals, according to [our] definitions, are ambiguous symbols,..., and have as many meanings as there are types.

Russell considers the solution of saying that two numbers have the same cardinality if there is a bijection between them (ibid.), but where does this correlation happen?

In order to make this correlation, he needs functions that apply across logical types.

In the end, Logicism in the strong form that mathematics just *is* logic came to be seen as too strong and unworkable.

Russell and Whitehead's treatment of the class paradoxes (as well as the language paradoxes and the Vicious Circle Principle, which popped up in the reading—we won't discuss it here to keep focussed on issues surrounding mathematics) via Simple Type Theory (this is the type theory we discussed in this seminar) and Ramified Type Theory (we haven't discussed this today, it concerns the admissibility of so-called *impredicative* definitions) would go on to pave the way for the work of scholars like Alonzo Church, who would make enormous contribution to the emerging field of theoretical computer science.

This is a recurring theme throughout these lectures: Philosophical ideas that don't necessarily work out sometimes have very significant (and indeed sometimes practical!) offspring.

5 Questions/Discussion

Question. Even if we allow that Logicism has the tools to interpret mathematics as part of logic, do such translations preserve *meaning*? For example, I can interpret each proposition about two-dimensional Euclidean space as about pairs of real numbers (via Cartesian coordinates). Do these propositions have the same *content* though? (Is this even a 'good' question to ask?)

Question. What do you think is the faulty principle at work in Russell's Paradox? (There is a bump in the rug, and we have to push it somewhere. With any paradox you can either (i) give up an assumption, (ii) deny a rule of inference used, or (iii) deny the conclusion is a problem. So, what should we say?)

Question. Do we think that the language paradoxes deserve the same treatment as the class paradoxes, or are they different?

Question. One question asked by several people concerned Frege's understanding of geometry, and whether it really is easier to think with the negation of (say) the parallels postulate rather than an arithmetical axiom. We can revisit this, if people are keen.