Week 5: Logicism: Frege

Introduction to the Philosophy of Mathematics
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Recap

Last week we saw:

1. The use of limits, an idea that seems to incorporate *some* notion of infinity, was important (especially in the work of Leibniz and Newton) for determining rates of change.

2. However, *inconsistent* suppositions about infinitesimals were made during computation.

3. While Newton and Leibniz were on the right lines, it seems that they did not have a fully coherent conception of the subject matter in question.

This week we'll see an attempt to get at the fundamental ideas underlying mathematics in the work of Frege.

1 Frege and the search for systematic foundations

Gottlob Frege (1848–1925) made several substantial contributions to philosophy (including the philosophy of language), logic, and mathematics (though his diary indicates that towards the end of his life he held some political views that we would now regard as pretty reprehensible).

We begin with Kant. Kant thought that the truths of arithmetic depended upon intuitions of temporal succession.[1]

This idea found its way into the philosophy of many 19^{th} century mathematicians and philosophers, e.g. Schröder.

Frege was very unimpressed with this 'psychologistic' interpretation of mathematics (that arithmetic is to be interpreted by reference to human psychology).

Frege (at least in his early work) clearly held a strong form of realism:

> Just as the geographer does not create a sea when he draws borderlines...so too the mathematician cannot properly create anything by his definitions. (Frege, Foreword to Volume I of the *Grundgesetze*)

Frege thought that numbers were assertions about *concepts*:

> If I say "the King's carriage is drawn by four horses", then I assign the number four to the concept "horse that draws the King's carriage". (Frege, *Grundlagen*, §46)

This is because:

1. We can make different number claims about the same things using different concepts (e.g. "here are four companies", "here are five-hundred soldiers").

2. Sometimes zero is the appropriate response (e.g. "The number of Venus' moons is zero").

[1] He thought in fact that they were *synthetic a priori*.

He also thought that numbers were self-subsistent entities:

> the individual number shows itself for what it is, a self-subsistent object. I have already drawn attention above to the fact that we speak of "the number 1", where the definite article serves to class it as an object. In arithmetic this self-subsistence comes out at every turn, as for example in the identity $1 + 1 = 2$. (Frege, *Grundlagen*, §57)

Here (and elsewhere) Frege claims that numbers are self-subsistent entities because:

1. They can figure into definite descriptions (e.g. "the number of toes on Neil's left foot is five").
2. They can be referred to by apparent singular terms.

Moreover, they are *abstract*:

> ...but where is the number 4? It is neither outside us nor within us...Not every object has a place. (Frege, *Grundlagen*, §61).

But he is sensitive to the following observation that has popped up constantly in our course:

> "How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them?" (Frege, *Grundlagen*, §62)

Frege's response to this problem is known as *logicism*.

2 Frege's Logicism

Frege's solution was to regard truths of arithmetic as *analytic* and knowledge of them as forming part of *logic*.[2]

[2]He is not alone in this. For example Dedekind seems to espouse some similar ideas at points, e.g.:

> In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number-concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate result of the laws of thought. ... It is only through the purely logical process of building up the science of numbers and by thus acquiring the continuous number-domain that we are ready to investigate accurately our notions of space and time (Dedekind, Preface to [Dedekind, 1888])

What grounds did he have for thinking this? There were two key ones:

(1.) We cannot adopt negation of a principle of arithmetic without confusion.

> For purposes of conceptual thought we can always assume the contrary of some one or other of the geometrical axioms, without involving ourselves in any self-contradictions...Can the same be said of the fundamental propositions of the science of number? Here, we have only to try denying any one of them, and complete confusion ensues. Even to think at all seems no longer possible. (*Grundlagen*, §14)

(2.) Statements of arithmetic apply to the widest possible domain.

> The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought? (Frege, *Grundlagen*, §14)

This was his strategy:

1. Set out a logically perfect language $\mathscr{L}$.
2. Set out definitions of arithmetical expressions in purely logical terms in $\mathscr{L}$.

3. Set out sentences in $\mathscr{L}$ which are clearly logically true, to be used as axioms; to set out clearly logically valid rules of inference for $\mathscr{L}$.

4. Explicitly derive the basic principles of arithmetic from these axioms by these rules.

Question. What, then for Frege, is a logically perfect language?

Answer. A language in which the semantic structure of a sentence is determined by its surface syntactic structure, and which has no ambiguity or vagueness.

In setting this out, he developed his *Begriffsschrift*; once of the first fully formalised systems of mathematics. (It's tricky to read, so we'll just work anachronistically with contemporary resources.)

Question. What, for Frege, is logic?

Answer:

1. First-order logic, plus...

2. Quantification over predicate variables as well as individual variables (known as second-order logic), plus...

3. A certain principle about concept-extensions (Basic Law V).

What is Basic Law V (BLV)?

This arises out of how he thought the second-order variables should be interpreted: as *concepts*.

> ...numbers are assigned only to the concepts, under which are brought both the physical and mental alike, both the spatial and temporal and the non-spatial and non-temporal. (Frege, *Grundlagen*, §48)

Basic Law V is then the logical law that pertains to these concepts, namely that they are *extensional*:

Definition. *Basic Law V* (or BLV) is the following claim (in more contemporary notation):

$\{ \ x \mid F(x) \ \} = \{x \mid G(x) \ \} \leftrightarrow [\forall x \ (F(x) \leftrightarrow G(x))]$

3 Hume's Principle and Arithmetic

Frege holds that we now have a *logically perfect language*, and laid down the *logical axioms* and *rules of inference*. We now have to show that arithmetic follows from these.

What do we mean by arithmetic? Here we are interested in the (second-order) Dedekind-Peano axioms (often referred to as the system $\mathbf{PA}_2$). Letting '0' be a constant to denote 0, '$\mathbb{N}(x)$' be a predicate for 'x is a natural number', and '$P(x, y)$' a predicate for 'x immediately precedes y' we have the following axioms :

(i) $\mathbb{N}(0)$

(ii) $\neg\exists x P(x, 0)$

(iii) $[\mathbb{N}(x) \wedge P(x, y)] \rightarrow \mathbb{N}(y)$

(iv) $[\mathbb{N}(x) \wedge P(x, y) \wedge P(x, y')] \rightarrow y = y'$

(v) $[\mathbb{N}(x) \wedge P(x, y) \wedge P(x', y)] \rightarrow x = x'$

(vi) $\mathbb{N}(x) \rightarrow \exists y P(x, y)$

(vii) **(Induction.)** $\forall F \ [(F(0) \wedge \forall x \ \forall y \ (\mathbb{N}(x) \wedge F(x) \wedge P(x, y)) \rightarrow F(y)) \rightarrow \forall \ x(\mathbb{N}(x) \rightarrow F(x))]$

So the challenge is to derive the axioms of $\mathbf{PA}_2$ from his axioms for second-order logic.

We won't go into the details of the derivation, but it includes the following steps:

(1.) Define a *membership predicate* $\in$.[3]

(2.) Use this, combined with the axioms of his system, to derive the following principle:

[3]This can be done roughly as follows: $x \in y \leftrightarrow_{df} \exists H[y = \{z|H(z)\} \wedge H(x)]$.

Comprehension Principle.[4]

$$F(y) \leftrightarrow y \in \{x|F(x)\}$$

(3.) **Bootstrapping.** 0 can then be defined as the class of all concepts with nothing in their extension (e.g. $x \neq x$), and more generally the number n can be defined as the extension of all extensions with exactly n instances.

(4.) Using these definitions and the Comprehension Principle, we then derive **Hume's Principle** or HP:

$$\sharp xFx = \sharp xGx \leftrightarrow F \approx G$$

(i.e. the number of Fs is the same as the number of Gs iff there's a bijection between them)

(5.) Then using HP you can derive the axioms of $\mathbf{PA}_2$ (it's a bit lengthy).

Added bonus. We get the existence of an infinite set for free! Just take:

$$\{x|\mathbb{N}(x)\}$$

4 Russell's Paradox and Frege's Response

OK success! Lunchtime!....sadly no.

The problem arises from the Comprehension Principle:

In Frege's system, it looks like some concept extensions are members of themselves whereas others are not. For example, the concept extension comprising all apples is not itself an apple, so that concept extension is not self-membered. The concept extension of all infinite extensions is (presumably) itself an infinite extension, and so should be self-membered.

Russell's Paradox. We consider the Russell Class $R = \{x|x \notin x\}$. By Comprehension we have $y \in \{x|x \notin x\} \leftrightarrow y \notin y$. We now substitute in R for y and obtain $\{x \mid x \notin x\} \in \{x \mid x \notin x\} \leftrightarrow \{x \mid x \notin x\} \notin \{ x \mid x \notin x\}$. Contradiction!

Informally. Is $R \in R$? If yes, then R violates its membership condition, and so $R \notin R$, contradiction. Therefore, $R \notin R$. But then R satisfies its membership condition, and so $R \in R$, contradiction!

Possible responses that Frege considers (under enormous pressure of a publication deadline, Russell alerted him to the problem whilst the second volume of the *Grundgesetze* was already in press):

(1) Some predicate extensions (classes) are not genuine objects (so you can't substitute $\{x|x \notin x\}$ in to object-position in the argument).

(2) There are no such things as predicate-extensions (again, so you can't substitute for object variables).

(3) Some predicates do not have extensions (so maybe $x \notin x$ doesn't have an extension at all).

(4) Basic Law V needs to be weakened.

(1) he rejects for two reasons: (a) formal complexity of the solution, and (b) how could we distinguish between the proper and improper objects?

(2) he rejects for the simple reason it would have been devastating to his project: Classes were essential to his logical outlook.

(3) he rejects for similar reasons to (1), how do we find out the right cases?

(4) is what Frege opts for (he weakens one direction of BLV)[5]. He had little faith in the solution (or at least thought it needed serious further scrutiny), and in any case, his modification doesn't work along two pretty devastating axes: (i) it breaks many of his proofs as they are stated, and (ii) it's still inconsistent (with mild assumptions).

Around 1906 Frege gave up on the reduction of arithmetic and logic, and reverted to a Kantian view.

[4] For a version of the derivation, see [Giaquinto, 2002], p. 57.

[5] See here [Giaquinto, 2002], p. 56.

5 Questions/Discussion

Lots of good questions again this week: Unfortunately we can't address them all (please email me/organise an office hour if you would like to discuss more).

Question. What do we think about Frege's initial reasons (generality, inability to conceive of things being otherwise) for accepting that arithmetic was just part of logic?

Question. What about the claim that number ascriptions are always relative to a concept?

Question. Does it matter if numbers are objects? Or could we get truth-value realism some other way? Is this even desirable?

Question. How are first-order resources (i.e. quantification into object position, e.g. $(\exists x)\phi(x)$ for some formula ϕ) and second-order resources (i.e. quantification into predicate position e.g. $(\exists F)F(x)$) different? Is second-order logic as precise as first-order logic?

Question. Despite the fact that his overall system was inconsistent, Frege's derivation of $\mathbf{PA}_2$ from HP is impeccable. Is this another case (following [Colyvan, 2008]) where an inconsistent theory is used 'acceptably' (in some sense)?

Question. Do we think that the laws of logic themselves are up for debate?

References

[Colyvan, 2008] Colyvan, M. (2008). Who's afraid of inconsistent mathematics? *Proto-Sociology*, 25:24–35.

[Dedekind, 1888] Dedekind, R. (1888). Was sind und was sollen die Zahlen? In *[Ewald, 1996]*, pages 787–832. Oxford University Press.

[Ewald, 1996] Ewald, W. B., editor (1996). *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, volume II. Oxford University Press.

[Giaquinto, 2002] Giaquinto, M. (2002). *The Search for Certainty: A Philosophical Account of Foundations of Mathematics*. Oxford University Press.