Week 4: The Ghosts of Departed Quantities? Berkeley and the Analysts

Introduction to the Philosophy of Mathematics
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Recap

Last week we saw a conception of infinity that was quite different from Plato's conception of numbers as independent objects.

Aristotle had a conception of mathematics on which mathematical objects don't exist unless the corresponding objects do.

He had a conception of infinity on which there is no such thing as *actual* infinity, but rather any infinity was *potential*.

This week, we're going to fast forward about 2'000 years.

We miss a lot of interesting material in the meanwhile.

For example: The development of Neo-Platonism by Plotinus (205–270 C.E.), the subsequent integration of these ideas into theological/Christian thought by St. Augustine (354–430 C.E.) and St. Thomas Aquinas (1224–1274 C.E.). all the while there was some fascinating developments in the Islamic world. (e.g. Thābit ibn Qurra al-Ḥarrānī (836–901 C.E.) seems to have had a version of Galileo's paradox.) We also see some beginnings of an understanding that infinite division with respect to geometry can create problems if thought about naively (e.g. Duns Scotus (1266–1308 C.E.) and circles of equal diameter).[1]

[1] For an overview of some of the history, see [Moore, 1990], Ch. 3.

1 A brief history and explanation of the calculus

This week we'll examine the *calculus*, the ideas of *infinity* contained therein, and the idea of *rates of change*.

This will be more difficult than the previous two with respect to mathematical content.

But nonetheless, I think it's important to know about:

1. It deals with *central* concepts in mathematics.
2. It shows a fulcrum point of *conceptual change*.
3. We shouldn't shy away from mathematical details!

So: Don't be too worried if you find the details difficult! I will summarise the key philosophical features as we go.

The first mathematical development that we need occurs well before the pioneering work in the calculus of Newton (1642–1727) and Leibniz (1646–1716).

René Descartes (1596–1650) and Pierre de Fermat (1601–1665) noticed that there were links between *numbers* and *geometry*.

They noticed that *points* in an n-dimensional geometric space (we'll stick to two or three dimensions) can be coded by n-tuples of numbers.

This is now a staple of high-school mathematics, and we take it for granted! But at the time the realisation opened up *radically* different proof-techniques.

A *function* $f : \mathbb{R} \to \mathbb{R}$ can then be understood by its *graph* and a *continuous function* (don't worry if you don't know what this is!) as a *line*.

If we then consider the modelling of physical objects in these spaces, we can ask questions about *rates of change* at points.

Easy enough for straight lines. But what about *curves*? We need the *tangent*.

2 Newton and Leibniz's Calculus

There is a question of priority or independence with Newton and Leibniz.

We'll focus on Newton, but I take no stand on the priority debate.

The early analysts' noticed that the slope of a line between two points depended upon the *difference quotient*.

For example, if the function we're considering is $f(x) = x^2$, and we take two points x and $x + i$, the difference quotient is:

$$(*)\frac{(x+i)^2 - x^2}{i}$$

We want the limit of this as i approaches 0.

The early analysts' purported solution was as follows: Treat i as a quantity smaller than any finite quantity, but greater than 0. As Newton explains:

> "Finite particles are not moments, but the very quantities generated by the moments. We are to conceive them as the just nascent principles of finite magnitudes..." ([Newton, 1687], pp. 261–262)[a]
>
> [a]For more, see Book I of Newton's *Principia* ([Newton, 1687], p.95).

We can then reason as follows:

(1.) By $(*)$ we have that $f'(x)$ is $\frac{x^2+2xi+i^2-x^2}{i}$ as i approaches 0.

(2.) Simplifying, we get $\frac{2xi+i^2}{i}$.

(3.) Dividing through by i yields $2x + i$.

(4.) But now note that i is smaller than any finite quantity when it approaches 0, so we can eliminate it, obtaining the result that the derivative at x of $f(x)$ is $2x$, as desired.

Question. Who can spot the problem?

3 Berkeley's criticisms

George Berkeley (1685–1753), Bishop of Cloyne (Ireland) wrote a treatise entitled:

The Analyst; or, A Discourse addressed to an Infidel Mathematician. Wherein it is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith. 'First cast out the beam out of thine own eye; and then shalt thou see clearly to cast out the mote out of thy brother's eye.'

For obvious reasons, I'll refer to this as *The Analyst*.

In *The Analyst*, Berkeley makes several arguments (some philosophical, some mathematical) criticising the analysts.

His point was not to completely discredit their work, but rather to defend theology against criticisms.[2]

We'll focus on three interlinked criticisms. The first concerns what fluxions (infinitesimals) are meant to be:

> And what are these Fluxions? The Velocities of evanescent Increments? And

[2]See, for example, §XX of *The Analyst*:

> It must be remembred that I am not concerned about the truth of your Theorems, but only about the way of coming at them; whether it be legitimate or illegitimate, clear or obscure, scientific or tentative.

what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities? ([Berkeley, 1734], §XXXV)

Berkeley argues here that there has not been a coherent description of infinitesimals given.

This links to a second point:

I have no Controversy about your Conclusions, but only about your Logic and Method. How you demonstrate? What Objects you are conversant with, and whether you conceive them clearly? What Principles you proceed upon; how sound they may be; and how you apply them? ([Berkeley, 1734], §XX)

Here he points out, that given that we have no coherent interpretation, we do not seem to be able to lay down hard and fast rules for how infinitesimals can be reasoned with. (Later in the course we'll discuss *axiomatisation*; the laying down of rules for reasoning (usually symbolic). There's a sense in which Berkeley anticipates this move, though his request for axioms is not for a formalised theory.)

The third point concerns specific moves made in reasoning:

Nothing is plainer than that no just Conclusion can be directly drawn from two inconsistent Suppositions. You may indeed suppose any thing possible: But afterwards you may not suppose any thing that destroys what you first supposed. ([Berkeley, 1734], §XV)

Here Berkeley argues that one cannot reason once one has arrived at a contradiction.

A modern classical perspective: Standard logics validate *explosion* or *ex false quodlibet*—from a contradiction you can derive anything!

4 Resolution

The resolution took a long time to emerge.

It requires a little mathematical sophistication, so we won't go into it too deeply (it emerges through the work of some great mathematicians such as Cauchy, Bolzano, and Weierstrass).

The core idea is to use alternation of quantifiers to *describe* the value of a function as it approaches a limit, without ever *actually* appealing to anything infinitely small.

Definition. $\lim_{x \to p} f(x) = l$ iff for all $\epsilon > 0$ there is a $\delta > 0$ such that for any x, if $0 < |x - p| < \delta$, then $0 < |f(x) - l| < \epsilon$.

Roughly put: Any time you challenge me (given some f and p) with a value ϵ, I can find a δ such that for any value of x, if $|x - p|$ is less than δ, then $|f(x) - l|$ is less than ϵ.

Philosophical Point. The (ϵ, δ)-definition of a limit talks about the *values* of the function that *exist* closer and closer to some value, rather than talking about infinitesimally small quantities.

5 Questions and two puzzles

Puzzle 1. [Colyvan, 2008] Let's accept that the early analysts' reasoning was inconsistent. Nevertheless, they applied their results with great success! Should we accept that usage of inconsistent mathematics is acceptable?

Puzzle 2. [Smith, 2015] As it turns out, using the modern conception of limits, there are *multiple* different ways we can cash out the derivative (e.g. standard derivative, symmetric derivative). It's not clear that Newton was thinking with either: He *lacked* the conceptual resources to distinguish them. Should we say that he was thinking with one or other, both, or neither?

Relevant: Newton's conception of the calculus was very different, it seems that he is talking about a physical-like space rather than a purely-mathematical one.

> These quantities [position, velocity and so on] I here consider as variable and indetermined, and increasing or decreasing, as it were, by a perpetual motion or flux. ([Newton, 1687], pp. 261–262)

Question. Do Berkeley and Leibniz's views about the calculus integrate into their wider philosophical views?

Question. Does Berkeley really succeed in defending theology with his *tu quoque*?

Question. Does the modern (ϵ, δ)-notion of a limit presuppose infinity?

Question. Can we save infinitesimals? (Answer: There are two modern theories: (1) *Smooth Infinitesimal Analysis* gives up classical logic and (2) *Robinson Analysis* constructs infinitesimals via a quite complicated ultrafilter construction. It's too technical to deal with in this course, but you are welcome to discuss these things with me in an office hour.)

References

[Berkeley, 1734] Berkeley, G. (1734). The analyst. London.

[Colyvan, 2008] Colyvan, M. (2008). Who's afraid of inconsistent mathematics? *Proto-Sociology*, 25:24–35.

[Moore, 1990] Moore, A. W. (1990). *The Infinite*. Routledge.

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