Week 3: Potential Infinity: Aristotle

Introduction to the Philosophy of Mathematics
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Recap

Last week we saw some views from Plato and the Pre-Socratics:

- Anaximander seems to have had a view on which the apeiron, which at least *looks* like a conception of infinity, was the origin (archê) of all things.
- The Pythagoreans tied themselves in knots of the seeming infinitude present in geometrical figures (in particular their realisation that $\sqrt{2}$ was irrational) and their belief that everything could be represented by (ratios of) natural numbers.
- Plato seems to have countenanced an infinite multitude of natural numbers, possibly both as Forms and some kind of other abstracta. Mathematical objects are:
  - Non-spatio-temporal and acausal.
  - Known through a faculty of 'recollection' (which we might now interpret as a kind of perception).

1 Some points about Aristotle's philosophy of mathematics

This week, we'll look at Aristotle (384 B.C.E.–322 B.C.E).

Aristotle was an incredible polymath, contributing to the natural sciences including physics and biology, but also was an accomplished ethicist, political theorist (and so on...).

Aristotle is notoriously difficult to interpret (and everything I say should thus be viewed as already providing my own interpretative lens).

However, he is more systematic (in a sense) than Plato, writing works that are meant to provide theories to which he is committed, rather than posing problems.

However, what we have was roughly in the form of lecture notes, and so shouldn't be treated as *definitive*.

Furthermore, Aristotle never wrote a treatise directed at the philosophy of mathematics, though as you've seen from the *Physics* he became interested in some close themes (especially *infinity*).

It seems though, that he regarded platonism[1] as a rather *bloated* ontology.

> "the Forms are, one may say, more numerous than perceptible particulars (though it was in seeking causes for the latter that they [the Platonists] went on from them to the Forms), because in each case there is, over and above the real objects, something else with the same name, both for things around us and for

[1] I'll follow [Annas, 1978] here in distinguishing Platonism (i.e. the views of Plato) from platonism (the philosophical view about ontology/metaphysics inspired from Plato, that we discussed in the last lecture).

eternal things." (*Metaphysics*, 1078b)

We see here that Aristotle identifies that Plato held that there are things *beyond* the properties that have them.

In particular, he considers a version of the Third Man Argument at *Metaphysics* 83–84. (Plato also discusses the Third Man Argument in the *Parmenidies*.)

There's (as with much ancient philosophy) a lot of controversy about how it should run. But here's one version[2]:

One-Over-Many. For any plurality of F things, there is a form of F-ness by virtue of partaking of which each member of the plurality is F.

Uniqueness For any property F, there is exactly one form of F-ness.

Self-Predication. For any property F, the form of F-ness is F.

Non-Self-Partaking. The F is not among the things partaking in the form of F-ness.

We can then generate a regress. Suppose some things xx are F (say large). Then (by One-Over-Many and Uniqueness) there is a unique form of F-ness in which they partake. But by self-predication, the form of F-ness is itself F. We now consider the plurality obtained by adding F to the xx (call this plurality yy). These are all F, and so must have a form F_1 in which they all partake. Also $F_1 \neq F$ by (Non-Self Partaking). Again, F_1 must be F, and so we consider the yy with F_1 added...etc...

This seems to be problematic because of the *epistemological* work Plato wanted the forms to do, they are supposed to *explain* how we have knowledge.

So Aristotle didn't think that the Forms were a good theory.

But he did think that numbers exist (at least in some sense):

> So since it is true to say without qualification not only that separable things exist but also that non-separable things exist (e.g. that moving things exist), it is also true to say without qualification that mathematical objects exist. (*Metaphysics*, 1077b)

...and that they were abstract: Not located in space and time etc.

> The mathematical branches of knowledge will not be about perceptible objects just because their objects happen to be perceptible, though not [studied] as perceptible; but nor will they be about other separate objects over and above these. (*Metaphysics*, 1078a)

Aristotle instead held that abstract objects only exist if the corresponding physical objects do.

How then do we know about mathematical objects for Aristotle?

His answer: *abstraction*.

> A man is one and indivisible as a man, and the arithmetician posits him as one indivisible, then studies what is incidental to a man as indivisible; the geometer, on the other hand, studies him neither as man nor as indivisible, but as a solid object. (*Metaphysics*, 1078a)

Question. What is this notion of abstraction?

The process of steadily ignoring properties of an object.

e.g. The geometer abstracts away from the imperfect shapes of the world to form perfect figures. The arithmetician abstracts away from objects to consider them as discrete entities.

Abstraction seems to be a process of the mind for Aristotle.

[2] I suppress some other premises here. See [**?**] for a nice summary.

In this way we see the roots of constructivism in Aristotle.

Platonism. Mathematical objects are mind-independent abstract entities.

Mental Constructivism. Mathematical objects are *constructions of the mind*.

It's not clear whether Aristotle held this.

But the first signs of this view (that I know of) appear in Aristotle.

2 Aristotle's arguments against actual infinities

Does Aristotle think that there is an *actual* infinity of mathematical objects?

Well, recall his philosophical view that mathematical objects need to be instantiated in the physical in order for them to exist.

The question then comes down to whether physical reality is infinite.

Some ancient Greeks held this, plausibly Anaximander, but also Anaxagoras (c. 510 – c. 428 BCE) (who held that everything was infinitely divisible in some sense) and the atomist Democritus (c. 460 – c. 370 BCE), who held that the atoms were infinite in number and variety.

Aristotle's answer is: **NO!**

He gives several arguments against the existence of infinity in the *Physics*.

However, we should first note how careful he is provide a definition of infinity.

> We must begin by distinguishing the various ways in which the term 'infinite' is used: in one way, it is applied to what is incapable of being gone through, because it is not its nature to be gone through (the way in which the voice is invisible); in another, to what admits of a traversal which cannot be completed, or which can only be completed with difficulty, or what naturally admits of a traversal but does not have a traversal or limit. (*Physics*, 204a3-204a6)

Here is a first isolation of what is infinite:

Aristotelian Definition. The infinite is that which cannot be *traversed* in a non-trivial sense (To clarify: A voice is invisible in a trivial sense, since it isn't the sort of thing that could be visible. Similarly there might be things that cannot be traversed because they are not of the right kind.)

He then considers several views of the infinite and dismisses each:

(1.) The infinite is a substance (he attributes this to Anaximander).

Aristotle's Verdict. *Absurd.* If it were a substance, it would have parts which are infinite, and he thinks that one infinity cannot be many infinities.

(2.) The infinite numbers a plurality (he attributes this to the atomists).

Aristotle's Verdict. *Absurd.* For Aristotle, a number is what can be arrived at by counting, and you can't count to an infinite number.

(3.) There is an actually infinite body in the world.

Aristotle's Verdict. *Empirically false.* Suppose by body we mean 'bounded-surface'. Then the result is trivial (since bounded). Suppose then that we have a more liberal idea of body.

- He rejected infinite divisibility on the grounds of Zeno's paradoxes.

- He rejected infinitely large on the grounds that the infinite element (fire, earth, air, water) would destroy the others. The arguments here are a bit puzzling; they depend on a very outdated metaphysics and it's unclear whether they work in this context. At the time though, they would have been state-of-the-art.

It seems then, that Aristotle's arguments against the infinite, while a substantial advance for their time, were not watertight. However, let's grant him that there is no physical body that is infinite.

3 Potential Infinity

Aristotle was sensitive to the fact that it is desirable to be *non-revisionists* about mathematical subject matter:

> Our account does not rob the mathematicians of their science... (*Physics*, Book III, 2207b28–207b34)

Consider now the following two views:

Mathematical Revisionism. It is acceptable to substantially revise mathematics on the basis of ontological considerations.

Mathematical Non-Revisionism. It is *not* acceptable to substantially revise mathematics on the basis of ontological considerations.

These distinctions will pop up throughout the course.

Given Aristotle's desires to interpret the work of the geometer and arithmetician, it seems that he was (something) of a non-revisionist, at least as far as arithmetic and geometry were concerned.[3]

He identifies five main sources of the infinite:

> Belief in the existence of the infinite comes mainly from five considerations: [1.] From the nature of time—for it is infinite; [2.] From the division of magnitudes—for the mathematicians also use the infinite; [3.] again, if coming to be and passing away do not give out, it is only because that from which things come to be is infinite; [4.] again, because the limited always finds its limit in something, so that there must be no limit, if everything is always limited by something different from itself. [5.] Most of all, a reason which is peculiarly appropriate and presents the difficulty that is felt by everybody—not only number but also mathematical magnitudes and what is outside the heaven are supposed to be infinite because they never give out in our thought. (*Physics*, Book III, 203b16-203b26)

He also isolates two kind of infinities:

Infinite by division. That which can be infinitely *divided*.

Infinite by addition. That which can always be *added* to.

He couples this with an account of *potentiality* vs. *actuality* (or 'fulfilment').

e.g. I am *potentially* movable, but can also be *actually* moving.

e.g. Some pieces of wood are *potentially* a box, but can also *actually* be a box.

Aristotle's response was that things are only *potentially* infinte, not *actually* infinite.

We can understand this (anachronistically) as a scope distinction. e.g. "This body is infinitely divisible."

Actual Infinite. There is a possible situation S, such that for every natural number n, the body has been divided n-many times.

Potential Infinite. For every natural number n, there is a possible situation S, such that the body has been divided n-many times.

His argument was that there are *potential* infinities, but not *actual* ones.

He then argues that all we need for mathematics is the *potential* infinite:

> Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do

[3]Though see [Annas, 1978] (pp. 39–40) for some discussion of rational and irrational numbers.

> not need the infinite and do not use it. They postulate only that a finite straight line may be produced as far as they wish. It is possible to have divided into the same ratio as the largest quantity another magnitude of any size you like. Hence, for the purposes of proof, it will make no difference to them whether the infinite is found among existent magnitudes. (*Physics*, Book III, 207b28–207b34)

4 Questions/Discussion

Question. Several people asked about Aristotle's use of the potential infinite and Zeno's paradoxes of motion. Is there a relationship here?

Question. A few people asked about how we might know infinity, especially when considerations of potential infinity seem to suggest we could only ever know a portion of it. Thoughts?

Question. Does the existence of potential infinite imply/presuppose an actual infinite?

Question. Is infinity a *property* or an *object* (or both)?

Question. A few questions asked about some mathematical details. Given the non-revisionist strand in Aristotle, we might ask whether we really *need* the actual infinity to do mathematics? (For the present day? For Aristotle? How might his notion of infinity be used in the contemporary context, e.g. with respect to real numbers? With respect to the composition of lines by points?)

References

[Annas, 1978] Annas, J. (1978). Aristotle's metaphysics, books m and n. *Philosophical Review*, 87(3):479–485.