Week 2: Plato and the Pre-Socratics

Introduction to the Philosophy of Mathematics
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1 Recap

Last week we saw (aside from the outline of the course:

(1.) Infinity crops up in lots of areas of mathematics.

(2.) Naive reasoning about infinity gets us into trouble easily.

This week, we're going to visit the earliest (Western) historical records of reasoning about infinity.

2 A Pre-Socratic: Anaximander

We begin with Anaximander (c. 610—546 B.C.E.).

Anaximander, like many of the ancients (e.g. Thales, Heraclitus, etc.), was concerned with the fundamental questions of metaphysics.

Core Question. What is the fundamental *nature* of the world around us? (The Greeks often put this in terms of the "*archê*" or "origin" of the Universe, which can be interpreted both literally and abstractly.)

He wrote what is possibly the oldest surviving fragment of Western philosophy:

> Whence things have their origin,
> Thence also their destruction happens,
> As is the order of things;
> For they execute the sentence upon one another
> —The condemnation for the crime—
> In conformity with the ordinance of Time.

What is Anaximander referring to when he talks about the "origin" of all things?

This is controversial! But one interpretation (as we have gleaned from certain passages from other scholars, e.g. Aristotle) is the notion of απειρον (which, for typesetting reasons, I'll use the latin text *apeiron* from now on).

Question. What is *apeiron*?

The notion of *peras* is normally translated as that which has *limit* or *bound*.

So *apeiron* is that *without* limit or bound.

At the time, this was probably quite controversial.

This is because one natural idea regarding the universe was that it was a celestially-bounded thing, a sort of metaphysical terrarium.

The ancients already disputed whether Anaximander had something clear in mind with *apeiron*.

We'll see some questions at the end of the seminar. Some immediate ones are:

Question. Does "origin" really refer to *apeiron* for Anaximander?

Question. Is the *apeiron* really infinite?

Question. In particular, he talks about *destruction* in the fragment. There therefore seems to be a notion of *process* at work. So is *apeiron* really *infinite* or is it just *inexhaustible* in some sense?

Question. Is it *physical*? Or *abstract*? Is it rather about *thought*, and limits there?

Question. Is the *apeiron* really a kind of *mathematical* infinite (if infinite)?

3 A Pythogorean Digression

Pythagoras lived c. 570 – c. 495 BC.

The *Pythagoreans* contrasted strongly with Anaximander.

Where Anaximander saw the nature of the world as essentially *boundless*, the Pythagoreans saw the motions of the planets etc. as essentially indicative of order and limits.

They regarded the *apeiron* as something abhorrent.

The Pythagorean Doctrine. The world can be represented by natural numbers (or ratios thereof).

But what about a right-angled triangle formed by taking the diagonal from the unit square.

As the Pythagoreans discovered, this triangle has hypotenuse $\sqrt{2}$, and $\sqrt{2}$ is *irrational* (that is, can't be represented by a ratio of natural numbers).

It looks like boundlessness is here to stay!

(According to legend, the Pythagoreans (or the Gods, depending on who you believe) drowned one of their fold (Hippasus of Metapontum) for discovering this (or possibly publicising it).)

Question. Does the diagonal of the unit square *really* commit us to *apeiron*?

4 Plato on number and infinity

We are left with many questions about the *apeiron*, and whether or not it was really mathematical.

One famous philosopher who *directly* addressed questions concerning the epistemology and metaphysics of mathematics was Plato (429?–347 B.C.E.).

He had his own conception of *apeiron*

which was more abstract, it seemed to be related to the idea of *indefiniteness*. (See the discussion in the *Philebus* for example.)

Let's proceed then by examining his philosophy of mathematics rather than his understanding of *apeiron*.

However, we have to be *careful*, Plato writes *dialogues*, and it is unclear whether he is actually espousing the views he presents, or just canvassing them for the sake of argument.

Question. I've given reading from Book VII of Plato's *Republic*, a text on *political philosophy*. Why?

The *Republic* is concerned with the topic of justice, and relates a discussion between Socrates and various interlocutors. A (far too condensed!) outline:

- Book I: They begin with a discussion of justice, but end in *aporia*.
- Book II: They continue with the discussion, and suggest looking at a city-state in order to know the concept better.
- Book III: They discuss how a guardian class should be raised and educated.
- Book IV: Socrates (among other things) discusses how the nature and virtues of the soul are analogous to what can be found in the ideal city.
- Book V: Socrates defends against some criticisms of his claims from Book III.
- Book VI: They examine the concept of the philosopher-king, and Socrates argues that they should have an understanding of the *good*.
- Book VII: Socrates explains an account of how we come to gain more **knowledge** and how this might figure into the education of the philosopher king.
- Book VIII: They discuss and criticise some other forms of government (before Socrates argues for a philosophical aristocracy).
- Book IX: They transfer some of the observations about cities to individual psyches.
- Book X: Provides a conclusion (and discusses some problems with imitative art, among other things).

The key point is that Socrates provides an account of how a person can obtain 'good' knowledge, and how knowledge is *structures* in the analogies of the *Sun*, *Line*, and *Cave*.

The Sun. Plato, in Book VI (508–509), speaks about the Sun as similar to the form of the good.

> Exactly.
> Then the sun is not sight, but the author of sight who is recognised by sight.
>
> True, he said.
> And this is he whom I call the child of the good, whom the good begat in his own likeness,

to be in the visible world, in relation to sight and the things of sight, what the good is in the intellectual world in relation to mind and the things of mind.

Will you be a little more explicit? he said. Why, you know, I said, that the eyes, when a person directs them towards objects on which the light of day is no longer shining, but the moon and stars only, see dimly, and are nearly blind; they seem to have no clearness of vision in them?

Very true.
But when they are directed towards objects on which the sun shines, they see clearly and there is sight in them?

Certainly.
And the soul is like the eye: when resting upon that on which truth and being shine, the soul perceives and understands and is radiant with intelligence; but when turned towards the twilight of becoming and perishing, then she has opinion only, and goes blinking about, and is first of one opinion and then of another, and seems to have no intelligence?

(*Republic*, Book VI, 508b–e)

The sun is that which allows for the illumination of other things. It is through closeness to the good (and intellectual) that other things become clear and precise.

The Analogy of the Line. The line for Plato (which first appears in Book VI) provides a kind of hierarchy for human thought. The parts of the line are (from top to bottom):

- Understanding (noesis) [Part of the 'intelligible' realm, seems to be knowledge of the forms, and right at the top, the good.]
- Thought (dianoia) [Also part of the 'intelligible' realm, this is where knowledge of 'mathematics' and the 'sciences' appears.]
- Belief (pistis) [Part of the 'visible' realm (this is where thought about physical objects lives).]
- Imagination (eikasia) [Part of the 'visible' realm (this is where thought about shapes/shadows lives).]

Where does mathematics lie? This is a source of great debate. Parts of what Plato says seem to

The Analogy of the Cave. This analogy, it seems, is partly there to bolster the discussion concerning the Line. We are to imagine prisoners chained to a wall, so that all they can see are shadows of figurines on the wall. A prisoner escapes, and first sees the *actual* figurines, which are still representations of the animals they represent. Going outside they are initially blinded by the light of the sun, and can only look at the shadows of the real objects. As they become acclimatised, they are able to look at the real objects, and (finally) the sun. (Note here the relationship with the Line!)

(Fun fact, later the prisoner returns to the others, and is persecuted for his beliefs by his fellow prisoners.)

Central these analogies is that the highest form of knowledge is knowledge of the *forms*.

The forms are abstract, non-spatio-temporal entities that ground the properties that objects have. Objects have properties by *participating* in the forms. For example, virtually all the clothing I wear at university participates in the form of *blackness*.

Part of the role of studying mathematics is to facilitate the journey of the would-be philosopher upward towards the forms.

But things aren't so simple! For example, Plato writes the following in the *Phaedo*:

> [Socrates speaks about simple arithmetic, as narrated by Phaedo:] And you would loudly exclaim that you do not know how else each thing can come to be except by sharing in the particular reality in which it shares, and in these cases you do not know of any other cause of becoming two except by sharing in Twoness, and that the things that are to be two must share in this, as that which is to be one must share in Oneness. (*Phaedo* 101b–e)

Aren't numbers *forms* then? Oneness, Two-ness, etc.?

A possible way out: There are the number *Forms* and numbers as other *abstracta*. [Note: This is a controversial position! See the Introduction, §1, of Annas' book for a thorough treatment.]

Problem. This seems to conflict with the claims that we want to make about mathematics.

e.g. There are exactly two natural numbers strictly between 2 and 5. There seem to be four (abstract-3, Form-3, abstract-4, Form-4).

Possible response. We should *relativise* our talk (which is ambiguous) between the abstract-numbers and the form-numbers. (This is a common strategy in the philosophy of mathematics, which we'll see more of later in the course.)

In any case, there are several facets of what we'll call *platonism* in the philosophy of mathematics:

> Visitor: Now then, we take all the numbers to be beings.
> Theatetus: Yes, if we take anything else to be.
> (*Sophist* 238a–b)

Platonism Claim 1. *Existence.* Mathematical entities exist and are objects (insofar as anything else is).

We can now ask, how many numbers are there?

> Parmenides: Do you think there is any number that need not be?
> Aristotle: In no way at all.
> Parmenides: Therefore, if one is, there must also be number.
> Aristotle: Necessarily.

> Parmenides: But if there is number, there would be many, and an unlimited multitude of being. Or doesn't number, unlimited in multitude, also prove to partake of being?
> Aristotle: It certainly does. (*Parmenides*, 144a)

Platonism Claim 2. Mathematical objects are unlimited in their number.

What, then, are these objects *like*?

> Now, no one with even a little experience of geometry will dispute that this science is entirely the opposite of what is said about it in the accounts of its practitioners.
> How do you mean?
> They give ridiculous accounts of it, though they can't help it, for they speak like practical men, and all their accounts refer to doing things. They talk of "squaring," "applying," "adding," and the like, whereas the entire subject is pursued for the sake of knowledge.
> Absolutely.
> And mustn't we also agree on a further point?
> What is that?
> That their accounts are for the sake of knowing what *always is, not what comes into being and passes away*.
> That's easy to agree to, for geometry is knowledge of what *always is*. (*Republic*, Book VII, 527a–b)

Platonism Claim 3. *Eternal.* Mathematical objects are eternal.

There are times though, when Plato comes *close* to suggesting that numbers are not eternal.

> "As my account has it, our sight has indeed proved to be a source of supreme benefit to us, in that none of our present statements about the universe could ever have been made if we had never seen any stars, sun or heaven. As it is, however, our ability to see the periods of day-and-night, of months and of years, of equinoxes and solstices, has led to the *invention* of number, and has given us the idea of time and opened the path to inquiry into the nature of the universe. (*Timaeus*, 47a)

We'll encounter this idea, that numbers are mental constructions, later in the course (when we discuss Brouwer). For now we'll put it aside.

Why might Plato think that mathematical objects are eternal. Well, perhaps because he thinks of mathematical objects as really outside space and time, as the discussion of the sun, line, and cave suggests.

Platonism Claim 4. *Abstractness.* Mathematical objects are *abstracta*, that is they are non-causal (in the physical sense), and not located in space and time.

This might also lead us to think that:

Platonism Claim 5. *Mind-Independence.* Numbers do not depend on us for their existence.

It is an open question, the extent to which Plato subscribed to all these clauses of platonism!

But the characterisation raises a deep epistemological challenge.

Benacerraf's Epistemological Challenge. Given that numbers are non-spatio-temporal, acausal entities, how do we gain knowledge of them?

A plausible account of Plato's response might be gleaned from his remarks in the Meno:

There, Socrates asks a slave boy several (some might say quite leading) questions concerning the diagonal of a square. Since the slave boy comes up with his responses on his own, Plato conjectures that the soul is 'recollecting' information from a time when it was able to encounter abstracta:

> As the soul is immortal, has been born often and seen all things here and in the underworld, there is nothing which it has not learned; so it is in no way surprising that it can recollect the things it knew before, both about virtue and other things. As the whole of nature is akin, and the soul has learned everything, nothing prevents a man, after recalling one thing only a process men call learning discovering everything else for himself, if he is brave and does not tire of the search, for searching and learning are, as a whole, recollection. (*Meno* 81c–e)

This perhaps seems to make Plato's account of mathematical epistemology dependent upon a somewhat dubious account of the soul. However, perhaps it can be made more palatable:

> [Socrates speaks:] But our present discussion, on the other hand, shows that the power to learn is present in everyone's soul and that the instrument with which each learns is like an eye that cannot be turned around from darkness to light without turning the whole body. This instrument cannot be turned around from that which is coming into being without turning the whole soul until it is able to study that which is and the brightest thing that is, namely, the one we call the good. (*Republic*, Book VII, 518c–d)

Here it seems that Plato suggests that we have a perception-like faculty, that allows us to reach out to abstracta.[1]

Note: Benacerraf would have no truck with such an argument.

There has been very little empirical evidence discovered to suggest that we do actually have such a faculty.

But perhaps we can gain knowledge of the abstract form of numbers by studying their instances.

An analogy: I can learn about the aural form of the opening riff to Led

[1] See also 525b, 527d–e.

Zeppelin's 'Black Dog' by listening to their recording, or any other recording for that matter.

There is some empirical evidence that we have some perceptual knowledge of *instances* of number, but that's a story for another day!

5 Questions

Great questions everyone! I've picked out a few here. Don't stress if one of yours wasn't picked, (i) there were some duplicates that I left out, and (ii) there were some fab questions that foreshadowed quite far ahead in the course and presupposed too much to be included at this stage (e.g. inconsistent multiplicities were mentioned).

Question. (1) Platon will die Philosophen und Politiker in der Rechenkunst ausbilden, um deren Fähigkeit zu wahrer Erkenntnis zu verbessern. Dabei müsse die Rechenkunst aber um ihrer selbst willen ausgeübt werden, und nicht zum Zwecke des Handels oder anderer praktischer Anwendungen. Lässt sich diese Auffassung aus heutiger Perspektive noch halten? Ich denke dabei insbesondere an die praktischen Wissenschaften wie die der Physik, welche sich der Mathematik als Handwerkszeug bedienen. Eröffnen diese uns nicht viel mehr Möglichkeiten, zur Erkenntnis des wahren Seins vorzudringen, als es die Mathematik allein je tun könnte?

Closely related: **Question.** Doesn't Plato contest the modern way of science (or the methods of science in general) as he states, that "crafts" which base their research on empty hypothesis can never lead to true knowledge? This is contradictory to our understanding of science, which exist to verify or falsify a theory. It seems like this method would indeed lead to knowledge.

Question. For Plato arithmetic and calculation lead us towards truth (525b) by studying and understanding the nature of numbers (525c). I guess he means that the knowledge conveyed by arithmetic and calcutlation is knowledge about the world of Ideas. Does this mean that for Plato there is an Idea of the infinte?

Question. Related to Anaximander: In Chapter 2 Couprie talks about different interpretations of „the boundless". I see the possible meaning as temporarily limitless as well as „something inexhaustible" or without qualifications. But no argument seems to follow the mentioned interpretation as spatially limitless?

Question. In §3a (Anaximander) it says: „The boundless has no origin. For then it would have a limit". If we think about the natural numbers they have a starting point: 1 (or 0) and then go on indefinitely. Does this mean that according to Anaximander the natural numbers are not infinite (e.g. not boundless)? But the integers are?

Question. Closely related: By adopting the vertical interpretation of Anaximander's conception of the Boundless, according to which everything that is is generated from the Boundless, one could question the exclusivity of the boundless property. Should only the initial, generating Boundless which could be regarded as a boundless entity or could we imagine that some other boundless forms of existence could flow out of it? That question could have a theological meaning, in the sense it could be linked to the greek conception of plural divinity, or even a mathematical one, aiming to determine if the Boundless could itself generate other boundless sets or theorical fields.

Question. Q1: Anaximanders „the boundless" clearly is a notion of infinity. What interests me is what kind of infinity it applies too. Potential or actual? In the 2nd Paragraph of the Encyclopedia is an interpretation from Anaximander himself, namely: "that which is inexhaustible". This seems to me as an interpretation as potential infinity on the one hand and of an interpretation of actual infinity on the other hand. Potential because you can add elements/items/things infinitively often. Actual because that "Inexhaustible" may existed as a whole, from the beginning. But can it be both?