Week 1: Some Short Remarks

Introduction to the Philosophy of Mathematics
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1 Motivation

Maybe you think that mathematics is just intrinsically interesting, and philosophical questions about it are in and of themselves worth pursuing.

But even if you don't, some really *useful* insights and technologies have emerged from the study of the philosophy of mathematics too.

Who has used a smartphone this morning?...Who is reading this on one **right now**?

At the heart of that machine are a bunch of instructions encoded into bits.

Part of the intellectual roots of the languages they encode can be traced back to something called *Type Theory*, originally developed by Russell and Whitehead in the early 1900s ([Russell and Whitehead, 1910]).

Interestingly, this didn't start out as an attempt to come up with a theory of *computing* per se (though some ideas of this kind come up as early as Leibniz and Pascal[1]) but rather to provide a *foundation* for mathematics.

(What is a *foundation* for mathematics? This is a difficult question (see, e.g. [Maddy, 2017]). Part of what we'll do in this course is try and introduce you to these ideas and questions. Roughly: To come up with a system of principles that can represent mathematics in a justified way.)

[1] See `https://en.wikipedia.org/wiki/Stepped_reckoner`.

2 Kinds of mathematics

What do we mean when we talk about *mathematics*?

Trivial answer: The stuff that mathematicians do.

Better elucidation: The study of different kinds of *subject matter*, e.g.

- Number theory:
  - The natural numbers: $0, 1, 2, 3, ...$
  - The integers: $0, 1, -1, 2, -2, 3, -3, ...$
  - The rational numbers: Anything you can represent with a fraction.
  - Real numbers: Anything you can represent with an infinite decimal (e.g. $1.\dot{0}...$, π, $\sqrt{2}$, $0.\dot{3}$).
- Geometry:
  - Study of different kinds of *space* and the *points*, *lines*, and *planes* that appear in them.
- Analysis:
  - The study of infinite limits (e.g. infinite sums, how the value of a function changes as it approaches a point)
- And far more besides...[2]

(Of course, these subject matters are interrelated, e.g. Cartesian coordinates and geometry.)

[2] Just check out `https://en.wikipedia.org/wiki/Areas_of_mathematics`.

Something that's interesting about those areas of mathematics is that many of the structures we're dealing with are infinite.

I've taken the approach in this course of thematically focussing on attitudes to the infinite, rather than giving you a broad selection of different topics. Each approach has its advantages, but mine is to try and get you to think deeply about a single theme, the hope being that you can then apply this to other areas of study.

3 Philosophical questions about the infinite

Philosophical Puzzle. How can we, as mere finite shaved apes, speak meaningfully and have knowledge about the infinite? (Are we finite? Does the 'infinite' *really* exist? If so, do we *really* have knowledge about it?)

Often in philosophy, it's useful to divide inquiry along two axes:

Metaphysics. What is the *nature* of the infinite?

- Is there such a thing?
- If so, is it independent from our minds?

Epistemology. How to we gain knowledge of the infinite?

- Do we, in fact, have knowledge?
- If so, how do we access something that is presumably so much bigger than ourselves?

As we'll see throughout the course, these two dimensions (epistemological/metaphysical) are often taken to be linked.

4 Some immediate challenges

Pre-theoretically, we might think that the notion of infinity is somewhat problematic. Here's a couple of classic problems:

Galileo's Paradox. Consider the natural numbers $0, 1, 2, 3, ..., n, n+1, ...$ Now consider the set of all squares of natural numbers $0, 1, 4, 9, ..., n^2, (n+1)^2, ...$ 'Clearly' there are 'more' natural numbers than squares of natural numbers, since every square is a natural number, but not vice versa. But also 'clearly' there are the 'same number' of squares as natural numbers, since we can correlate the squares one-to-one with the natural numbers with the function $f(n^2) = n$.[3]

Achilles and the tortoise. (This example is basically derived from Zeno, though probably in a different form from his original statement (his original work doesn't survive).) Suppose that Achilles runs much faster than a tortoise (let's call her "Tortuga"). Tortuga claims that, so long as she gets a head start, Achilles will never catch her. Note that, to catch Tortuga, Achilles will have to reach the place where Tortuga started. In the intervening period though, Tortuga will have run a little further. So Achilles must now get to Tortuga's new position. But in the time it takes him to cover this new distance, Tortuga will have moved a little further away (...and so on...). It seems that Achilles must accomplish infinitely many tasks in order to overtake Tortuga, but normally we don't think we can do infinitely many things in a finite amount of time. But clearly we *do* think that Achilles will catch Tortuga, and given some appropriate speeds and distances it would be relatively easy to calculate.[4] So, what is going on here?

5 Course structure

I'll leave it there for introductory material, hopefully that will have whet your appetite for learning about the philosophy and mathematics of the infinite.

[3] Something philosophers do a lot is use 'scare quotes' when they either (a) don't mean literally what they said, or (b) acknowledge that their use of a term is problematic.

[4] See Aristotle's *Physics* VI:9, 239b15. Zeno used these arguments to claim that motion is impossible, in service of Parmenides' contention that there is just one thing.

As I said earlier, the approach I have taken in the course is to get you to think thematically about the philosophy of mathematics of the infinite. I think this is good, since once you've learnt how to think deeply about a subject, you can then apply this way of learning to other areas. It does mean we're leaving out a lot though, including:

1. Explanation in mathematics.
2. The indispensability arguments and the role of mathematics in science.
3. Nominalism about mathematical objects.
4. Fictionalism.
5. Structuralism.
6. Visual thinking in mathematics.
7. Philosophy of mathematical practice.
8. Eastern/Islamic thought about infinity and its mathematics.
9. Many more...

Remarks on difficulty. Because we are going deep into some material this will be a *difficult* course, even if it's introductory in flavour. Do not be discouraged! Some remarks:

1. There is some more philosophical material, and some material that will talk about mathematics. You can write your work on whatever you want, and I will bear in mind the difficulty of the course when marking essays.
2. There is nothing wrong with not following certain parts of the course (this is philosophy, it's meant to be hard!). Making it easy would cheapen its value.
3. Ask questions! We have plenty of time and I will say if I need to move on.

References

[Caicedo et al., 2017] Caicedo, A. E., Cummings, J., Koellner, P., and Larson, P. B., editors (2017). *Foundations of Mathematics: Logic at Harvard Essays in Honor of W. Hugh Woodin's 60th Birthday*, volume 690 of *Contemporary Mathematics*. American Mathematical Society.

[Maddy, 2017] Maddy, P. (2017). Set-theoretic foundations. In *[Caicedo et al., 2017]*, pages 289–322. American Mathematical Society.

[Russell and Whitehead, 1910] Russell, B. and Whitehead, A. (1910). *Principia Mathematica to *56*. Cambridge University Press.