The Hyperuniverse Programme Tutorial II: Philosophical Issues Surrounding the Hyperuniverse

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Recap

- Last time, we saw several axioms, and several mathematical challenges from the Hyperuniverse Programme.
- This time, we'll examine some philosophical issues (that motivate some additional theorems).

But before we begin...

"A stubborn geometer might insist—like an exotic-travelogue writer who never actually ventures west of seventh avenue—that only Euclidean geometry is real and that all the various non-Euclidean geometries are merely curious simulations within it. Such a position is self-consistent, although stifling, for it appears to miss out on the geometrical insights that can arise from the other modes of reasoning. Similarly, a set theorist with the universe view can insist on an absolute background universe V, regarding all forcing extensions and other models as curious complex simulations within it...Such a perspective may be entirely self-consistent, and I am not arguing that the universe view is incoherent, but rather, my point is that if one regards all outer models of the universe as merely simulated inside it via complex formalisms, one may miss out on insights that could arise from the simpler philosophical attitude taking them as fully real." ([Hamkins, 2012], p426)

Introduction

1. Extensions. Seriously!?
    1.1 How to code them.
    1.2 Why consider them anyway?
2. Interpreting extensions of V.
3. Relationships with large cardinals and definable determinacy.
4. More open questions.

§1 Extensions

- We begin by recalling the Absolutist and Potentialist positions.
- And recalling the following two axioms:

The Inner Model Hypothesis

Let ϕ be a parameter free sentence of **ZFC**. Then V satisfies the Inner Model Hypothesis (or just the IMH) iff whenever ϕ is true in a width extension of V, then ϕ is true in an inner model of V.

$\sharp$-Generation Axiom

The $\sharp$-generation axiom states that V is the lower part of a model $\mathfrak{N}_\infty$ in the iteration of a sharp (N, U).

§1 Extensions

- Potentialism and Actualism seem to face complementary problems.
- The Actualist doesn't have the required extensions available, so the $\sharp$-Generation Axiom seems trivially false and the IMH seems trivially true.
- The Potentialist on the other hand has the extensions, but has to explain why a universe containing more sets is less maximal than one containing fewer sets.
- This latter objection against the Potentialist falls a bit flat, it ignores how her concept of set interacts with the her ontology.
- As we saw yesterday, the response for the Actualist takes a little more work.

§1 Extensions

A challenge and a constraint:

The Hilbertian Challenge.

Provide philosophical reasons to legitimise the use of extra-V resources for formulating axioms and analysing intra-V consequences.

The Methodological Constraint.

In responding to the Hilbertian Challenge, do so in a way that accounts for as much as possible of our naive thinking about extensions and links them to structural features of V. In particular, if we wish to apply an extending construction to V, there should be an actual set-theoretic model, resembling V as much as possible, that has an extension similar to the one we would like V to have.

§1 Extensions

- I'll go through the idea of V-logic in a little more detail.
- Two steps:
  1. Explain the system of V-logic and how it relates to satisfaction in extensions and $Hyp(V)$.
  2. Describe how to code $Hyp(V)$ using impredicative class theory.

§1 Extensions

What we'll do is very analogous to the use of the forcing relation:

Absolute-*MA*

We say that $\mathfrak{M}$ satisfies Absolute-*MA* iff whenever $\mathfrak{M}[G]$ is a generic extension of $\mathfrak{M}$ by a partial order $\mathbb{P}$ with the countable chain condition in $\mathfrak{M}$, and $\phi(x)$ is a $\Sigma_1(\mathcal{P}(\omega_1))$ formula (i.e. a first-order formula containing only parameters from $\mathcal{P}(\omega_1)$), if $\mathfrak{M}[G] \models \exists x \phi(x)$ then there is a y in $\mathfrak{M}$ such that $\phi(y)$.

Definition

We say that V satisfies Absolute-$MA^{\Vdash_{\mathbb{P}}}$ iff whenever $\mathbb{P} \in V$ is a partial order with the countable chain condition in V, and $\phi(x)$ is a $\Sigma_1(\mathcal{P}(\omega_1))$ formula, if there is a $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}} \exists x \phi(x)$, then there is a y in V such that $\phi(y)$.

§1 Extensions

Definition.

$\mathscr{L}_{\in}^{V}$ is the language consisting of:

(I) A predicate $\bar{V}$ to denote V.

(II) A constant $\bar{x}$ for every $x \in V$.

§1 Extensions

Definition.

V-logic is a system in $\mathscr{L}_{\in}^{V}$, with consequence relation $\vdash_V$ that consists of the following 'axioms':

(i) $\bar{x} \in \bar{V}$ for every $x \in V$.

(ii) Every literal (i.e. atomic or negated atomic sentence) of $\mathscr{L}_{\in} \cup \{\bar{x} | x \in V\}$ true in V is an axiom of V-logic.

(iii) The usual axioms of first-order logic in $\mathscr{L}_{\in}^{V}$.

For a set of sentences $\mathbf{T} \subseteq \mathscr{L}_{\in}^{V}$, V-logic contains the following rules of inference:

(a) Modus ponens: From $\mathbf{T} \vdash_V \phi$ and $\mathbf{T} \vdash_V \phi \rightarrow \psi$ infer $\mathbf{T} \vdash_V \psi$.

(b) The Set-rule: From $\mathbf{T} \vdash_V \phi(\bar{b})$ for all $b \in a$ infer $\mathbf{T} \vdash_V \forall x \in \bar{a}\phi(x)$.

(c) The V-rule: From $\mathbf{T} \vdash_V \phi(\bar{b})$ for all $b \in V$, infer $\mathbf{T} \vdash_V \forall x \in \bar{V}\phi(x)$.

§1 Extensions

- Proof codes in V-logic are (possibly infinite) well-founded trees with root the conclusion of the proof.
- Whenever there is an application of the V-rule, we get proper-class-many branches extending from a single node.

§1 Extensions

- How to code satisfaction in outer models?
- Introduce a new predicate $\bar{W}$ into the language, and introduce the following 'axioms':

'Axiom'.

$\bar{W}$-Width Axiom. $\bar{W}$ is a universe satisfying *ZFC* with the same ordinals as $\bar{V}$ and containing $\bar{V}$ as a proper subclass.

'Axiom'.

$\bar{W}$-Φ-Width Axiom. $\bar{W}$ is such that Φ.

'Axiom'.

$\Phi^{\vdash_V}$-Axiom. The theory in V-logic with the $\bar{W}$-Width Axiom and $\bar{W}$-Φ-Width Axiom is consistent under $\vdash_V$.

§1 Extensions

Definition.

$\bar{W}$-$\sharp$-Width Axiom. $\bar{W}$ is such that it contains a sharp that generates $\bar{V}$.

Axiom.

The Sharp Axiom$^{\vdash_V}$. The theory in V-logic with the $\bar{W}$-Width Axiom and $\bar{W}$-$\sharp$-Width Axiom is consistent under $\vdash_V$.

Axiom.

IMH$^{\vdash_V}$. Suppose that ϕ is a first-order sentence. Let **T** be a V-logic theory containing the $\bar{W}$-Width Axiom and also the $\bar{W}$-ϕ-Width Axiom (i.e. $\bar{W}$ satisfies ϕ). Then if **T** is consistent under $\vdash_V$, there is an inner model of V satisfying ϕ.

§1 Extensions

- *Hyp*(V) is the least admissible (i.e. model of **KP**, a very weak set theory) containing V as an element.

Fact.

[Barwise, 1975] (with a little tinkering to the current case of V) If ϕ has a proof in V-logic, then ϕ has a proof code in *Hyp*(V).

§1 Extensions

- How to code $Hyp(V)$?
- Brace yourselves...

§1 Extensions

Definition.

[Antos and Friedman, F] A pair $\langle M_0, R\rangle$ is a coding pair iff M_0 is a class with distinguished element a, and R is a class binary relation on M_0 such that:

(i) $\forall z \in M_0 \exists! n$ such that z has R-distance n from a (i.e. for any element z of M_0, z is a single finite R-distance away from a), and

(ii) let $\langle M_0, R\rangle \restriction x$ denote the R-transitive closure below x. Then if $x, y, z \in M_0$ with $y \neq z$, yRx, and zRx, then $\langle M_0, R\rangle \restriction y$ is not isomorphic to $\langle M_0, R\rangle \restriction z$, and

(iii) if $y, z \in M_0$ have the same R-distance from a, and $y \neq z$, then for all v, $vRy \rightarrow \neg vRz$, and

(iv) R is well-founded.

§1 Extensions

Definition.

[Antos and Friedman, F] Quotient Pairs. Let $\langle M_0, R\rangle$ be a coding pair and a be a set in M_0. We then define the equivalence class of a (denoted by '$[a]$') of all top nodes of the associated coding tree isomorphic to the subtree $\mathbb{T}_a$:

$[a] = \{b \in M_0 | \text{“}\langle M_0, R\rangle \upharpoonright b \text{ is isomorphic to } \langle M_0, R\rangle \upharpoonright a\text{”}\}$

Since we have Global Choice, we let $\tilde{a}$ be a fixed representative of $[a]$. We then define the *quotient pair* $\langle \tilde{M}_0, \tilde{R}\rangle$ as follows:

$\tilde{M}_0 = \{\tilde{a}| \text{ “}\tilde{a} \text{ is the representative of the class } [a] \text{ for all } a \in M_0\}$

$\tilde{a}\tilde{R}\tilde{b}$ iff "There is an $a_0 \in [a]$ and a $b_0 \in [b]$ such that $a_0 R b_0$."

§1 Extensions

Theorem.

[Antos, B., Friedman] Letting $(V)^+$ denote the structure of the quotient coding pairs over V, there is a coding pair tree for $Hyp(V)$.

- Σ^1_1-Comprehension for classes is sufficient.
- Δ^1_1-Comprehension for classes is not enough.

§1 Extensions

- Back to the philosophy...
- For the Potentialist and Actualist different things are said by the axioms.
- On the Actualist picture, these absoluteness principles link description and possibility.
- For the Potentialist, there is this dimension, but also a closure principle relative to other existent objects.
- This says something about how we should think of maximality and our concept of set on the Potentialist framework.

§2 Interpreting extensions of V

- So far so good for the Hilbertian Challenge.
- But what about the Methodological Constraint?
- We have no guarantee of any V-resembling models getting extended.

§2 Interpreting extensions of V

- Aim: Pull back truth about $Hyp(V)$ into V, and into the hyperuniverse.
- Several options here (in decreasing order of strength required):
- Solution 1: Reflection.
- Solution 2: Truth predicates.
- Solution 3: Feferman's trick.
- Solution 4 (informal): Brute avowal of the existence of Skolem functions.

§3 Relationship with Large Cardinals and Determinacy

- Let's move away from the metamathematical worries...
- If the IMH is true there are no inaccessibles in V, and the reals are not closed under $\sharp$.
- What about large cardinals and the very deep relationships with axioms of definable determinacy?
- There's a sense in which nothing is contradicted: There are some pictures consistent with large cardinals and determinacy.
- The Hyperuniverse Programme is methodologically pluralist in this sense, even if added to an ontologically monist framework.
- However, let's focus on the case where we think the IMH is accepted as true.

§3 Relationship with Large Cardinals and Determinacy

- What do we want large cardinals for?
- Indexing consistency strength in a linearly ordered fashion.
- With large cardinals in inner models we get a lot of this.
- Perhaps there are questions about exactly what the elements of this linear order are, and what is needed for the linear ordering.

§4 Open questions and future research

1. Precision regarding exactly what is required for the linear order.
2. Analysis of the interrelation between width and height?
3. Role and justification of strong class theories and infinitary logics.
4. Fusion with other programmes?

Thanks! Discussion!

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Antos, C., Barton, N., and Friedman, S.-D. (S).
Universism and extensions of V.
Submitted.

Antos, C. and Friedman, S.-D. (F).
Hyperclass forcing in Morse-Kelley class theory.
Forthcoming in *Journal of Symbolic Logic*.

Barwise, J. (1975).
Admissible Sets and Structures: An approach to definability theory.
Springer.

Hamkins, J. D. (2012).
The set-theoretic multiverse.
The Review of Symbolic Logic, 5(3):416–449.