The Hyperuniverse Programme Tutorial I: Maximality Through Absoluteness

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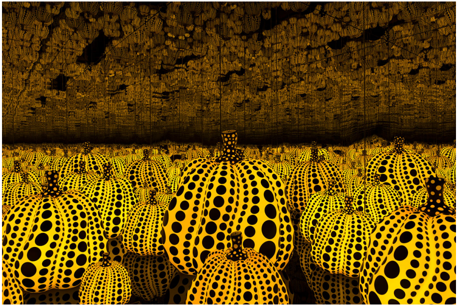
Setting the scene

- We know that in set theory there's a lot of independence from **ZFC**.
- How to tackle this?
- Apply maximality ideas, with a particular twist of absoluteness.
- What do we get?



—Yayoi Kusama and Matt Groening

Introduction

Plan for the tutorials:

1. Explain some of the axioms and results associated with the Hyperuniverse programme.
2. Discuss some of the surrounding philosophical issues.

Aim.

To present broad ideas and get discussion going. Please ask questions!

Introduction

Plan for this tutorial:

1. The main idea: Maximality through absoluteness.
2. Maximising height: $\sharp$-generation.
3. Maximising width: The Inner Model Hypothesis.
4. Synthesis?
5. Some open problems.

§1 Maximality Through Absoluteness

Consider the following two principles:

Second-order reflection

Let $\phi(A)$ be a formula in second-order set theory in the (second-order) parameter A. Then second-order reflection is the claim that, if $V \models \phi(A)$, there is a β such that $V_\beta \models \phi^\beta(A)^\beta$.

Martin's Axiom

Let κ be a cardinal such that $\kappa < |\mathcal{P}(\omega)|$. $MA(\kappa)$ is the claim that for any partial order $\mathbb{P}$ in which all maximal antichains are countable (i.e. $\mathbb{P}$ has the countable chain condition), and any family $\mathcal{D}$ of dense sets of $\mathbb{P}$ such that $|\mathcal{D}| \leq \kappa$, there is a filter F on $\mathbb{P}$ such that for every $D \in \mathcal{D}$, $F \cap D \neq \emptyset$. Martin's Axiom is then the claim that $\forall \kappa < |\mathcal{P}(\omega)|$, $MA(\kappa)$.

Question: What do these two principles have in common?

Absolute-MA

We say that $\mathfrak{M}$ satisfies Absolute-MA iff whenever $\mathfrak{M}[G]$ is a generic extension of $\mathfrak{M}$ by a partial order $\mathbb{P}$ with the countable chain condition in $\mathfrak{M}$, and $\phi(x)$ is a $\Sigma_1(\mathcal{P}(\omega_1))$ formula (i.e. a first-order formula containing only parameters from $\mathcal{P}(\omega_1)$), if $\mathfrak{M}[G] \models \exists x \phi(x)$ then there is a y in $\mathfrak{M}$ such that $\phi(y)$.

Core Idea.

The Hyperuniverse Programme aims to use this idea of absoluteness to motivate axioms capturing maximality.

- However, there's a twist.
- We're going to use extensions of V in capturing this maximality.
- For now we'll just work uncritically with extensions (we'll talk about this in detail next tutorial).
- For now, if you're feeling metamathematically queasy, just work in the Hyperuniverse: The collection of all countable and transitive models of **ZFC**.
- Please also assume it's a relatively well-populated place.

§2 Maximising Height: Sharp Generation

We first need two definitions:

Height extensions

Let V_1 and V_2 be universes. Then V_1 is a height extension of V_2 iff every member of V_2 is in V_1, and every V_α of V_1 is a V_α in V_2.

Width extensions

Again, let V_1 and V_2 be universes. Then V_1 is a width extension of V_2 iff every member of V_2 is a member of V_1, and they both have the same ordinals.

§2 Maximising Height: Sharp Generation

- We want absoluteness in height.
- We can take a popular perspective: Reflection.

§2 Maximising Height: Sharp Generation

- Step 1 (Standard Reflection): $V_{\kappa_0} \prec V_{\kappa_1} \prec ... \prec V_{\kappa_\infty} = V$
- Step 2 (Reverse Reflection):
  $V_{\kappa_0} \prec V_{\kappa_1} \prec ... \prec V_{\kappa_\infty} \prec V_{\kappa_{\infty+1}} \prec V_{\kappa_{\infty+2}} \prec ...$
- Step 2 has a problem that wasn't an issue for Step 1.
- The choice of sequence above V_{κ_∞} is not canonical.
- Can we make it canonical?

§2 Maximising Height: Sharp Generation

- Well, we want the universes to really resemble each other.
- So suppose I have some some $(V_{\kappa_{i_1}}, V_{\kappa_{i_2}})$ (with $i_1 < i_2$) and think of this structure as $(V_{\kappa_{i_2}}, \in)$ with a unary predicate for $V_{\kappa_{i_1}}$.
- We want any two pairs $(V_{\kappa_{i_1}}, V_{\kappa_{i_2}})$ and $(V_{\kappa_{j_1}}, V_{\kappa_{j_2}})$ to satisfy the same sentences, even allowing parameters that belong to both $V_{\kappa_{i_1}}$ and $V_{\kappa_{j_1}}$.

§2 Maximising Height: Sharp Generation

- Step 3 (Monster Chain): V occurs in a continuous elementary chain $V_{\kappa_0} \prec V_{\kappa_1} \prec ... \prec V_{\kappa_\infty} \prec V_{\kappa_{\infty+1}} \prec V_{\kappa_{\infty+2}} \prec ...$, of length $\infty + \infty$, where the models V_{κ_i} are strongly indiscernible in that for any n, and any two increasing n-tuples $\vec{i} = i_1 < i_2 < ... < i_n$ and $\vec{j} = j_1 < j_2 < ... < j_n$, the structures $V_{\vec{i}} = (V_{\kappa_{i_n}}, V_{\kappa_{i_{n-1}}}, ..., V_{\kappa_{i_1}})$ and $V_{\vec{j}}$ satisfy the same first-order sentences, sallowing parameters from $V_{\kappa_{i_1}} \cap V_{\kappa_{j_1}}$.

§2 Maximising Height: Sharp Generation

- What about higher-order?
- There are familiar challenges here (3-order reflection is inconsistent!).
- We take a familiar route, use embeddings to track the parameters.
- Again though, there's an extension-involving twist (in comparison to the very strong forms of reflection, e.g. Magidor, Welch, Bagaria, Roberts).

§2 Maximising Height: Sharp Generation

Sharps

A structure $\mathfrak{N} = (N, U)$ is called a sharp with critical point κ, a sharp, or just a $\sharp$, iff:

(I) $\mathfrak{N}$ is a model of **ZFC**$^-$ (i.e. **ZFC** with the power set axiom removed) in which κ is the largest cardinal and is strongly inaccessible.

(II) (N, U) is amenable (i.e. $x \cap U \in N$ for any $x \in N$).

(III) U is a normal measure on κ in (N, U).

(IV) $\mathfrak{N}$ is iterable in the sense that all successive ultrapowers starting with (N, U) are well-founded, providing a sequence of structures (N_i, U_i) and corresponding Σ_1-elementary iteration maps $\pi_{i,j} : N_i \longrightarrow N_j$ where $(N, U) = (N_0, U_0)$.

♯-generation

A model $\mathfrak{M} = (M, \in)$ is sharp-generated (or just ♯-generated) iff there is a sharp (N, U) and an iteration $N_0 \longrightarrow N_1 \longrightarrow N_2$... such that $M = \bigcup_{\alpha \in On^{\mathfrak{M}}} V^{N_\alpha}_{\kappa_\alpha}$ (V is the 'lower part' of $\mathfrak{N}_\infty$).

§2 Maximising Height: Sharp Generation

- This sharp generation assures us that V occurs as part of an indiscernible chain.
- The chain is canonical.

Fact.

If V is sharp generated, there are lots of indiscernible cardinals in V.

Fact.

If $0^\sharp$ exists, then so do $\sharp$-generated universes (e.g. L!)

§3 Maximising Width: Inner Model Hypotheses

- There's a challenge for trying to do the same for width.
- We don't have a clear iterative structure as we do for height.
- We'll just try stating principles outright, and seeing what we get.

§3 Maximising Width: Inner Model Hypotheses

The Inner Model Hypothesis

Let ϕ be a parameter free sentence of **ZFC**. Then V satisfies the Inner Model Hypothesis (or just the IMH) iff whenever ϕ is true in an inner model $I_{V'}$ of an outer model V' of V, then ϕ is already true in an inner model I_V of V.

§3 Maximising Width: Inner Model Hypotheses

Some interesting consequences:

Fact.

The IMH implies that $V \neq L$.

Theorem.

[Friedman et al., 2008] The IMH implies that there are inner models with measurable cardinals (of arbitrarily high Mitchell order).

Theorem.

[Friedman et al., 2008] The consistency of the IMH follows from the consistency of a Woodin cardinal with an inaccessible above.

§3 Maximising Width: Inner Model Hypotheses

- Two issues here:
  1. Introducing parameters.
  2. Synthesising with large cardinals.

§3 Maximising Width: Inner Model Hypotheses

Fact.

Allowing parameters into ϕ in the IMH results in an immediate contradiction.

The Strong Inner Model Hypothesis.

A parameter p is globally absolute if some formula defines it in all outer models which preserve cardinals up to and including the hereditary cardinality of p. Then SIMH(p) for a globally absolute parameter p states that if a sentence with parameter p holds in an outer model which preserves cardinals up to the hereditary cardinality of p then it holds in an inner model. The full Strong Inner Model Hypothesis (or SIMH) states that this holds for every globally absolute parameter p.

§3 Maximising Width: Inner Model Hypotheses

Theorem.

The SIMH implies that the continuum is massive (greater than $\aleph_\alpha$ for any α countable in L!).

Theorem.

[Friedman et al., 2008] The SIMH implies the existence of an inner model with a strong cardinal.

Fact.

It is not known whether the SIMH is consistent. However, restricted forms (e.g. to just ω_1-preserving extensions) are consistent relative to large cardinals.

§4 Synthesis?

Theorem.

The IMH implies that there are no inaccessible cardinals in V.

The IMH♯.

The IMH♯ is the *IMH* where we only allow ♯-generated width extensions.

Theorem.

Assuming the consequences of large cardinals (that every real has a ♯), the IMH♯ is consistent.

§5 Open problems

Some open problems before we close:

1. Are the SIMH and related principles consistent relative to large cardinals?
2. How might we motivate the restriction to parameters in the SIMH (and other variations)?
3. Does moving to a SIMH$\sharp$ affect its consistency?
4. Can we motivate the restriction to $\sharp$-generated universes with the IMH$\sharp$?
5. Is there an iterative or approximation story to be told about width (or indeed the hierarchy as a whole)?

Next time...

Lots of philosophical meat (with a little mathematical sauce) concerning:

- Being more critical about the use of extensions.
- How we might use the Hyperuniverse to render extension-talk more naturally.
- What happened to all the beautiful large cardinals (and why was I allowed to use them in this tutorial)?

Thanks! Discussion!

Hugely grateful to:

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Friedman, S.-D., Welch, P., and Woodin, W. H. (2008).
On the consistency strength of the inner model hypothesis.
The Journal of Symbolic Logic, 73(2):391–400.