Independence and Maximality: Lecture 3–Inner Model Reflection

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But first…

Since many people have provided some nice historical analysis, I thought I'd start with some history:

> *"In 1938 Lebesgue was given an honorary degree in Lwów, and he was taken to the coffee shop where the famous Polish mathematician Stefan Banach used to work. The waiter handed him a menu with long descriptions in Polish. Lebesgue glanced at it and answered, "Thank you, I only eat well-defined objects."." — Loren Graham and Jean-Michel Kantor, Naming Infinity, Belknap Press, Cambridge, MA, 2009.*

- This has nothing to do with my talk.
- Polish-mathematician-themed anecdotes very welcome in the pub later.

Aims for this lecture

- §1 Some preliminary remarks and a first attempt
- §2 Friedman's Inner Model Hypothesis
- §3 Variants of the IMH
- §4 Philosophical and mathematical conclusions

§1 Some preliminary remarks and a first attempt

- We saw last time that we could reflect down from V to initial segments to get some large cardinal consequences.
- In this way, we explored motivations for saying that the set-theoretic structures with which we work are rich or maximal in a certain respect.
- Recall, however, how we defined the iterative conception. There are two key parameters:
  1. How far the ordinals extend.
  2. What subsets get formed at successor stages.
- Reflecting down, certainly in the consequences it yields, seems to respond more to (1.).
- Can we have a more directly width-based concept of reflection?

§1 Some preliminary remarks and a first attempt

- There's a challenge here however.
- In the context of reflecting down there's an obvious candidate for what we should reflect to (namely V_αs), and they are well-ordered.
- It looks like for width we want to reflect to inner models, but there's no obvious corresponding way of 'generating' V by thickening inner models.
- Also V is standardly viewed as an inner model of itself, whereas V is not regarded as a rank initial segment of itself.
- N.B. Principles of this kind tend to assert the existence of inner models. These are proper classes, and so the principles are generally greater than first-order.

§1 Some preliminary remarks and a first attempt

- Despite these challenges, we'll try and take the bull by the horns and reflect statements inwards to inner models.
- Especially philosophically interesting here will be the use of extensions of models in formulating principles about the models themselves.

§1 Some preliminary remarks

Definition.

- Let ϕ be a first-order formula with/without first-order parameters
- (Inner Model Reflection) "If ϕ is true in V then it is true in a proper inner model of V"

The principle has some consequences:

Theorem.

Inner Model Reflection implies that $V \neq L$ (if parameters are allowed, we then get infinitely many inner models of V and that $V \neq L[x]$ for any set).

- I came up with this principle a week ago (I think it's likely the idea has been thought of before), so there's a non-trivial possibility that it's junk. Both versions are consistent relative to **ZFC** (h/t Andrés Caicedo and Joel Hamkins via `http://math.stackexchange.com`).
- Certainly this principle (even with parameters) will not be able to touch CH as it follows from the existence of unboundedly many measurable cardinals.

§2 Friedman's Inner Model Hypothesis

- It's hard to go beyond these first steps, however.
- The problem (it seems to me anyway) is that we've actually got very little information about V on the basis of its **ZFC** satisfaction.
- Notice also that a naive second-order version of the principle is either just flat-out inconsistent (with a 'full' interpretation of the second-order variables anyway), or it's very hard to see how to interpret the second-order variables.
- We need some way of adding juice to the principle.
- Here we find a role for extensions.

§2 Friedman's Inner Model Hypothesis

Leave any metamathematical baggage at the door for one second (if you spot the elephant in the room here, well done):

Definition.

A universe V^* is a width-extension of V iff it has the same ordinals as V, and $V \subset V^*$. [N.B. This is not just restricted to set-forcing!]

Definition.

[Friedman, 2006] The Inner Model Hypothesis (or just IMH) is the following statement (letting ϕ be parameter-free first-order):

> *"If ϕ is true in an inner model I^{V^*} of a width-extension V^* of V, then ϕ is true in an inner model I^{V} of V"*

§2 Friedman's Inner Model Hypothesis

We have consistency relative to large cardinals...:

Theorem.

[Friedman et al., 2008] The IMH is consistent relative to the existence of a Woodin cardinal with an inaccessible above.

...and some strength:

Theorems.

- $\mathbf{ZFC} + \text{IMH} \vdash V \neq L$.
- [Friedman et al., 2008] The IMH implies that there is an inner model with lots of mesaurable cardinals (in fact a proper class thereof).
- [Friedman, 2006] $\mathbf{ZFC} + \text{IMH} \vdash \neg\text{PD}$.

- Note that this means that the IMH has some anti-large cardinal consequences.

§2 Friedman's Inner Model Hypothesis

However:

Theorem.

[Friedman, 2006] $\mathit{ZFC} + \mathsf{IMH} \nvdash \mathsf{CH}$ and $\mathit{ZFC} + \mathsf{IMH} \nvdash \neg\mathsf{CH}$

Worse:

Theorem.

[Friedman, 2006] $\mathit{ZFC} + \mathsf{IMH} \vdash$ "There are no inaccessible cardinals in V"

§3 Variants of the IMH

We now have two technical challenges before us:

Challenge I.

Formulate something IMH-like that catches CH.

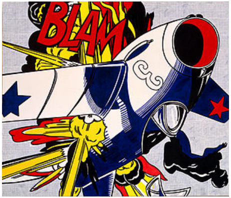
Challenge II.

Formulate a version of the IMH consistent with (at least some) large cardinals.

- Even better if we can fulfill these two desiderata simultaneously.
- Challenge II gets very technical, so we'll focus on Challenge I here. Please ask in discussion if you are interested.

§3 Variants of the IMH

One way of getting extra strength would be to introduce parameters into the IMH:



Theorem.

[Friedman, 2006] The IMH for arbitrary ordinal parameters is inconsistent.

Theorem.

[Friedman, 2006] The IMH for arbitrary real parameters is inconsistent.

§3 Variants of the IMH

We need a method for restricting the parameters:

Definition.

A parameter p is globally absolute iff there is a parameter-free formula which has p as its unique solution in all outer models V^* of V such that V and V^* have the same cardinals $\leq |TC(p)|$.

We then have:

Definition.

The Strong Inner Model Hypothesis (or just 'SIMH') is the statement that, for ϕ containing a globally absolute parameter p, if ϕ is true in an inner model I^V* of an outer model V^* of V, then ϕ is true in an inner model I^V of V.

§3 Variants of the IMH

Theorem.

- The SIMH implies that there is an inner model with a Strong cardinal.

Better:

Theorem.

- The SIMH implies that CH is false, and radically so: it cannot be $\aleph_\alpha$ for any α countable in L.

Sadly:

Fact.

It is unknown whether the SIMH is consistent relative to large cardinals.

§4 Philosophical and mathematical conclusions

There's a big metamathematical elephant in the room:

§4 Philosophical and mathematical conclusions

- All this time we've been using extensions of V!
- But what if (as many people think) V just is all the sets.
- Then the IMH is true, but utterly trivial.
- There's then a challenge, if you think that there's one Universe of sets: How to code extensions?
- Normally, the IMH and its variants are formulated about countable transitive models.
- I won't go into it here, but there's an interesting relationship between coding extensions and class theory (co-authored paper with Antos and Friedman soon to be submitted). [Feel free to ask in discussion though]

§4 Philosophical and mathematical conclusions

- There's also the question for the SIMH of what motivates the restriction on parameters.
- N.B. We had a similar problem here with Tait's reflection.
- One interesting philosophical project is to examine what uses of parameters are acceptable.
- This might change between philosophical view of the subject matter of set theory (e.g. Radical Potentialist vs. Universist).
- We might also wonder if there are other ways of capturing Inner Model Reflection.

§4 Philosophical and mathematical conclusions

- There's a deeper worry here though.
- Each of height and Inner Model Reflection seems to correspond to some way of capturing a maximality property of V.
- But they seem in conflict with one another.
- Are there ways of deciding between the two kinds of reflection?
- Or capturing the best of both?

§4 Philosophical and mathematical conclusions

- A blunt instrument of an argument would be to just point to how essential large cardinals are to set-theoretic practice, and thereby appeal to the priority of height reflection.
- Indeed, the consistency of statements like the IMH was argued for on the basis of large cardinals.
- Underlying this is an assumption that what we want from (or can be justified regarding) large cardinals is their truth in V.
- But a lot of the time we just need them for consistency proofs and the construction of inner models.
- This does not require their truth.
- Again, I'm not a large cardinals denier, but I do think that a deeper philosophical analysis than we currently have is merited (especially given the situation with Reinhardt cardinals).

Conclusions

- In this series of lectures, we started by examining large cardinals.
- We noted that there's problems with inconsistency and a limitation of what can be accomplished, and we might look for different ways of motivating a reduction in incompleteness.
- One way would be to examine reflection ideas as a precification of the maximality of set theory.
- There's some very deep mathematical and philosophical questions to be examined here!
- Incurvati, Luca - Maximality Principles in Set Theory. Forthcoming in *Philosophia Mathematica*.

I'll leave you with...

A beautiful quotation which has nothing to do with maximality or my lectures:

> *"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."—John von Neumann*

Dziękuję! Discussion!

Friedman, S.-D. (2006).
Internal consistency and the inner model hypothesis.
Bulletin of Symbolic Logic, 12(4):591–600.

Friedman, S.-D., Welch, P., and Woodin, W. H. (2008).
On the consistency strength of the inner model hypothesis.
The Journal of Symbolic Logic, 73(2):391–400.