Independence and Maximality: Lecture 2–Reflecting Down

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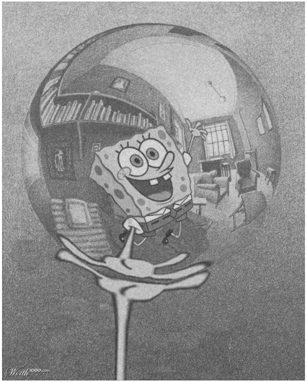
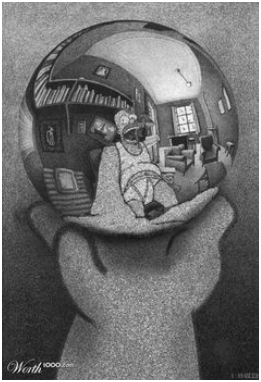
But first...





Structure of Lecture 2

- §1 Basic reflection principles
- §2 Third-order reflection and Tait's modification
- §3 Global reflection
- §4 Satisfaction predicates

§1 Basic reflection principles

Let's start with the following quotation from Gödel:

> *"The Universe of sets cannot be uniquely characterized (i.e. distinguished from all its initial segments) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal number." (Gödel in [Wang, 1977])*

§1 Basic reflection principles

The idea is that the intended set-theoretic structure(s) have so many sets of diverse kinds that anything true of V is also true of an initial segment thereof. More formally:

Template.

- Let ϕ be a sentence in some language, S be a variable over set-sized entities (of some kind), and ϕ^S denote the restriction of ϕ to S. Then a reflection principle states that:
- $\phi \rightarrow \exists S \phi^S$
- i.e. If ϕ is true in V, then ϕ is true when restricted to some set-sized object.

§1 Basic reflection principles

What's a really natural universe-y looking thing that's a set-sized part of V? An initial segment V_α:

Definition.

Let ϕ be a formula in the language of **ZFC**. Then first-order reflection is the following principle:

$(RP1)$ $\forall\alpha\exists\beta > \alpha\, \phi \leftrightarrow \phi^\beta$

This isn't very strong...

Theorem.

[Montague-Lévy] ($RP1$) is equivalent (modulo **ZC** – Infinity) to **ZFC**.

...but it does show that some reflection is part of our concept of set (insofar as **ZFC** is).

§1 Basic reflection principles

Notice that in our template of reflection principles, there were two main parameters:

1. The kinds of ϕ we allow.
2. The kind of thing to which we reflect.

§1 Basic reflection principles

What about if we allow ourselves higher-order formulas and parameters?

Definition.

Let ϕ be a formula of $\mathbf{ZFC_n}$, ϕ be a formula of $\mathbf{ZFC_n}$, and $A^{(n)}$ be a parameter of order n.
(RPn) $\phi(A^{(n)}) \rightarrow \exists\alpha\phi^{\alpha}(A^{(n),\alpha})$

- Here ϕ^{α} is the restriction of first-order quantifiers of ϕ to V_{α} and parameters.
- n^{th}-order quantification interpreted over $V_{\alpha+(n-1)}$ (or an isomorphic copy thereof if you want to keep track of the set-class-(α-hyperclass) distinction).
- Parameters are dealt with inductively:
  - A second-order parameter $A^{(2),\alpha}$ is interpreted as $A^{(2)} \cap V_{\alpha}$.
  - A higher-order parameter $A^{(m+1),\alpha}$ is interpreted as $\{B^{(m),\alpha} | B^{(m)} \in A^{(m+1)}\}$.

§1 Basic reflection principles

Example.

$(RP2)$ $\phi(A^{(2)}) \rightarrow \exists\alpha(V_\alpha, \in, V_{\alpha+1}) \models \phi(A^{(2,\alpha)})$

This principle has some nice consequences:

Theorem.

$\mathbf{ZFC_2} + (RP2) \vdash$ "There are unboundedly many inaccessible cardinals"

However, it's relatively weak:

Theorem.

$\mathbf{ZFC_2} + (RP2) \nvdash V \neq L$

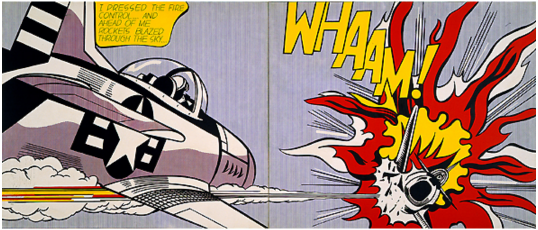
§2 Third-order reflection and Tait's modification

Okay let's increase the order of our formula to third-order reflection:

Definition.

$$\phi(A^{(3)}) \rightarrow \exists \alpha (V_\alpha, \in, V_{\alpha+1}, V_{\alpha+2}) \models \phi(A^{(3,\alpha)})$$

What do we get out of this principle?



§2 Third-order reflection and Tait's modification

Theorem.

[Reinhardt, 1974]/[Tait, 2005] ($RP3$) is inconsistent.

- Maybe there are philosophical reasons to be suspicious of third-order resources though? (e.g. the Universe view with some appropriate notion of proper class).
- Note that in this proof, the problem is that one is not guaranteed to have a substructure, just because $X^{(3)} \neq Y^{(3)}$ does not mean that there has to be any witnesses to this in V (similarly for $X^{(2)} \notin Y^{(3)}$).
- Tait's idea was thus to avoid the use of such formulas (i.e. ones contain $X^{(n)} \notin Y^{(n+1)}$ and $X^{(n)} \neq Y^{(n)}$).

§2 Third-order reflection and Tait's modification

Definitions.

- [Tait, 2005] A formula in a language of finite higher-order is positive iff it is built up from $\vee, \wedge, \forall$, and $\exists$, and atomic sentences of the form $x = y$, $x \neq y$, $x \in y$, $x \notin y$, $x \in Y^{(2)}$, $x \notin Y^{(2)}$, and (for $m \geq 2$) $X^{(m)} = Y^{(m)}$, $X^{(m)} \in Y^{(m+1)}$.
- The class of formulas $\Gamma_m^{(n)}$ is the class of positive formulas of the following form:
- $\forall X_1^{(n)}, \exists Y_1^{(k_1)}, \forall X_2^{(n)}, \exists Y_2^{(k_2)} ... \forall X_m^{(n)}, \exists Y_m^{(k_m)}$ $\phi(X_1^{(n)}, Y_1^{(k_1)}, X_2^{(n)}, Y_2^{(k_2)} ... \forall X_m^{(n)}, \exists Y_m^{(k_m)}, ..., A^{(l_1)}, ..., A^{(l_{n'})})$
- $\Gamma_m^{(n)}$-Reflection is then the (schematic) statement that if $V \models \phi$, for $\phi \in \Gamma_m^{(n)}$, then there is a $V_\gamma \models \phi^\gamma$.

§2 Third-order reflection and Tait's modification

Again we get some strength:

Theorem.

Suppose $V_\kappa \models \Gamma^{(2)}_m$-Reflection. Then κ is m-ineffable (a kind of partition cardinal).

- Recall our challenge from last time: Motivate the existence of large cardinals with a well-motivated restriction to ensure consistency.
- This is all starting to look a bit gerrymandered.
- No matter...

§2 Third-order reflection and Tait's modification

The proofs of the next two statements are a bit fiddly so we won't prove them here.

Theorem.

[Koellner, 2009] All of Tait's reflection principles are either consistent with $V = L$ (i.e. weak) or inconsistent. More precisely:

1. $\Gamma_n^{(2)}$-Reflection is consistent relative to the existence of an ω-Erdős cardinal (it's a particular cardinal defined by partition properties, $\kappa \longrightarrow (\omega)_2^{<\omega}$). This cardinal is consistent with $V = L$.
2. $\Gamma_1^{(3)}$-Reflection is inconsistent.

- We will need another technique to get a version of reflection that results in a substantial reduction of incompleteness. But first...

§2.5 Interlude: Koellner's Challenge

This introduces an interim challenge for the friend of reflection principles:

> *"the Erdős cardinal $\kappa(\omega)$ appears to be an impassable barrier as far as reflection is concerned. This is not a precise statement. But it leads to the following challenge: Formulate a strong reflection principle which is intrinsically justified on the iterative conception of set and which breaks the $\kappa(\omega)$ barrier."*
> *([Koellner, 2009], p217)*

Given the issues we've seen, some assurance that the principle is consistent is also desirable.

§3 Global reflection

Philip Welch and Leon Horsten, instead of postulating a formula-by-formula reflection, advocate a global resemblance between V and some V_κ:

> *"We have simply declared that the whole universe $(V, \in, \mathcal{C})$ is so rich that there is some κ so that the collection of parts over V_κ, namely $V_{\kappa+1}$, is in turn sufficiently rich so that any sentential truth we can formulate about the realm V with all of its parts (in the given language) reflects to a truth about V_κ with all of its parts." ([Welch, 2014], p16)*

§3 Global reflection

More formally:

Definition.

[Welch, 2014] (*GRP*) There is a non-trivial elementary embedding j and ordinal κ with $crit(j) = \kappa$ such that:
$j : (V_\kappa, \in, V_{\kappa+1}) \longrightarrow (V, \in, \mathcal{C})$

- Rather than reflecting formulas down to particular V_κ, this postulates that there is a class-by-class resemblance between some V_κ (with all its classes) and V (with all its classes).
- We then have:

Theorem.

$\mathbf{ZFC}_2 + (GRP) \vdash$ "There is a proper class of measurable Woodin cardinals."

§3 Global reflection

Also:

Definition.

A cardinal κ is α-extendible iff there is an ordinal θ such that κ is the critical point of a non-trivial elementary embedding $j : V_{\kappa+\alpha} \longrightarrow V_{\theta}$.

Theorem.

NBG + (GRP) is consistent relative to the existence of a 1-extendible cardinal.

§3 Global reflection

- There's lots to be said about the justification for (*GRP*).
- One worry is that it goes well beyond the idea of a 'reflection' principle.
- Essential is not just a formula-by-formula reflection, but also countenancing the existence of the embedding.
- But why should we accept that this exists?
- This isn't a fully-fledged objection, but rather a call to arms—there's interesting work to be done in fleshing out the philosophical account here.

§4 Satisfaction predicates

- Instead of using an embedding, we might look for a similar idea more in the spirit of the original idea of reflection.
- [Roberts, S] uses a second-order satisfaction predicate to generate the embeddings with more standard notions of reflection, rather than postulating the existence of embeddings outright:

§4 Satisfaction predicates

Definitions.

1. We can code collections of classes by using slices, the slice of X coded on y is the class $(X)_y = \{z | \langle y, z \rangle \in X\}$. [N.B. This is a useful trick outside of the discussion of reflection!].
2. We then say that a class X is a member of a class Y iff $X = (Y)_z$ for some $z \in dom(Y)$.
3. We say that a class X codes a set-sized collection of classes iff there is an x such that x is coextensive with $dom(X)$.
4. A class X is standard for a set x iff for all $y \subseteq x$, there is some $z \in dom(X)$ such that $(X)_z \cap x = y$.

Definitions.

- $Sat_2(x, X)$ is a predicate that holds iff the second-order formula coded by x is satisfied by the variable assignment coded by the class X.
- $\mathbf{ZFC_2(S)}$ is then the theory obtained by augmenting the language of $\mathbf{ZFC_2}$ with $Sat_2(x, X)$, adding the Tarski biconditionals, and admitting the use of $Sat_2(x, X)$ into the Class Comprehension Scheme of $\mathbf{ZFC_2}$.
- We let (SR_S) be the principle in the extended language that $\phi \rightarrow \exists\alpha\exists X($"$X$ is set-sized" $\wedge$ "X is standard for V_α" $\wedge\, \phi^{\alpha,X})$.

§4 Satisfaction predicates

We then have a following two theorems:

Theorem.

[Roberts, S] (SR_S) implies that there is a proper class of 1-extendible cardinals.

Theorem.

[Roberts, S] (SR_S) is consistent relative to the existence of a 2-extendible cardinal.

§4 Satisfaction predicates

- The work here is still very fresh, and it will take some time for the mathematical dust to settle.
- However, there's a lot of interesting work to be done.
- One issue is how we should conceive of second-order satisfaction, and how we should interpret the second-order variables.
- However, and this is a somewhat subtle point, we still get a lot out of Roberts' principle in the absence of the use of a full second-order satisfaction predicate.
- Another is to examine the ways in which (GRP) and (SR_S) can be made inconsistent, and to see whether their restrictions are natural.

Conclusions

- We've seen in this lecture various ways of trying to capture the idea that each of our favourite set-theoretic structures should be indistinguishable (in some sense) from one of their initial segments.
- There are various ways of making this idea precise.
- There is some possibility for extendability to inconsistency.
- In any case, formula-by-formula reflection is a little weak.
- There's much philosophical work to be done in examining global-style reflection principles (e.g. extendability to inconsistency, possible motivations for restrictions, detailed motivation for the principles, metamathematical issues...).

I'll leave you with

"People kept telling me "Less is more". I always said "How can that be? How can less be more? It's impossible! More is more!"."— Yngwie Malmsteen

Thanks! Discussion!

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