Independence and Maximality: Lecture 1
Independence and Large Cardinals

Neil Barton
Kurt Gödel Research Center

7th September 2016

First: A Quotation

> *"There are two ways to do great mathematics. The first is to be smarter than everybody else.The second way is to be stupider than everybody else – but persistent."—Raoul Bott*

Please be persistent and ask questions throughout!

Aim for the three lectures

The overall aim is to examine how conceptions of maximality in set theory interact with the independence phenomenon. In particular, we'll analyse this through the lens of reflection:

1. In this first lecture we'll look at two kinds of independence, various characterisations of large cardinals, and their relation to independence.
2. In lecture two we'll examine large cardinal reflection principles.
3. In lecture three we'll explore inner model reflection and internal consistency.

Structure of Lecture 1

- §1 Techniques relating to independence
- §2 Large cardinals and some preliminary reduction of incompleteness
- §3 More reduction and a limitation
- §4 Extendability to inconsistency
- §5 Some philosophical remarks

§1 Techniques relating to independence

- The early days of naive set theory were beset by paradoxes (e.g. Russell's Paradox, Cantor's Paradox, and the Burali-Forti Paradox).
- The iterative conception of set provides a story why the problematic 'collections' fail to form sets...
- ...and also gives us a nice visual representation of the set-theoretic structure(s) with which we work, explaining what we take to be a standard model (it should have the 'real' $\in$, be well-founded, be transitive etc.).

§1 Techniques relating to independence

We'll start with my favourite mathematical theorem:

Theorem.

[Cantor] For any set x, there is no bijection between x and $\mathcal{P}(x)$ [in particular there is no bijection between ω and $\mathcal{P}(\omega)$].

- This was the theorem that made me want to study set theory as an undergraduate.
- This is especially so as it implies (in combination with Replacement) that there's a vast cardinal structure of sets to be explored on the iterative conception.

An immediate natural question:

Question.

[Cantor] Given that there's no bijection between ω and $\mathcal{P}(\omega)$, is there a set intermediate in cardinality between ω and $\mathcal{P}(\omega)$? If so, how many?

- Cantor's Continuum Hypothesis (or just 'CH' for short): No. The next cardinal after $\aleph_0$ is $|\mathcal{P}(\omega)|$ (in other words, $2^{\aleph_0} = \aleph_1$).

§1 Techniques relating to independence

Unfortunately (or fortunately, depending on how you look at it) CH can't be resolved on the basis of the standard **ZFC** axioms:

Theorem.

[Gödel, 1940] Assuming that there is a model of **ZFC**, then there is a model L of **ZFC** such that $L \models$ CH.

How to do this:

Definition.

$L_0 = \emptyset$
$L_{\alpha+1} = Def(L_\alpha)$, for successor $\alpha + 1$
$L_\lambda = \bigcup_{\beta<\lambda} L_\beta$
$L = \bigcup_{\alpha \in On} L_\alpha$

So **ZFC** $\nvdash \neg$CH.

§1 Techniques relating to independence

Theorem.

[Cohen, 1963] Suppose there is a (transitive) model of **ZFC**. Then there is a model $\mathfrak{M}$ of **ZFC** such that $\neg$CH.

To show this we use a technique called forcing:

- You take some (carefully chosen) partial order $\mathbb{P}$ in your model $\mathfrak{M}$.
- Then grab a filter G on $\mathbb{P}$ that intersects every dense set of $\mathbb{P}$ in $\mathfrak{M}$.
- With a clever choice of names and method for evaluating these names, you add the new set G to $\mathfrak{M}$.
- Effectively, this adds G to $\mathfrak{M}$ and closes under the operations definable in $\mathfrak{M}$.
- You can use forcing to add κ many reals for virtually any κ, producing a model $\mathfrak{M}[G]$ where $\neg$CH holds.
- So **ZFC** $\nvdash$ CH.

§1 Techniques relating to independence

Even worse:

Theorem.

Let $\mathfrak{M}$ be any (countable, transtive) model of **ZFC**, then $\mathfrak{M}$ has forcing extensions $\mathfrak{M}[G]$ (collapsing no cardinals) and $\mathfrak{M}[H]$ (adding no new reals) such that:
$\mathfrak{M}[G] \models \neg \mathrm{CH}$
$\mathfrak{M}[H] \models \mathrm{CH}$

So, we seem to fix very little cardinal structure on the basis of **ZFC** alone.

Definitions.

- Consider ω-length two-player games of perfect information.
- At the end of a play of such a game we have generated a real $r \in {}^{\omega}\omega$.
- Now take some $X \subset {}^{\omega}\omega$. Say Player I wins if $r \in X$ and Player II wins if $r \notin X$.
- A game can be thought of as a tree, whose winning condition is a subset of ${}^{\omega}\omega$.
- A strategy is a function that tells a player what to play next at a finite stage of the game.
- A game is determined iff one of the two players has a winning strategy (i.e. there is a function that will tell them how to play at every finite stage of the game such that they never lose employing this strategy).

§1 Techniques relating to independence

Can we think of any determined games?

1. How about when $X = \emptyset$?
2. Or when $X = {}^{\omega}\omega$?
3. What if X is countable?

§1 Techniques relating to independence

A determinacy axiom states that for some class Y of $X \subseteq {}^{\omega}\omega$, every game whose winning condition is one of the Y is determined. For example:

Definition.

The Axiom of Determinacy States that every game is determined. It is inconsistent with **ZFC** (in particular AC).

Definition.

Borel Determinacy is the claim that every Borel set of ${}^{\omega}\omega$ (don't worry if you don't know what this means—it's just some topologically defined subsets of ${}^{\omega}\omega$) is determined. Borel Determinacy follows from **ZFC**.

Definition.

Projective Determinacy is the claim that every projective set of reals is determined (again, don't worry if you don't know the definition of projective set—it's an extension of Borel). It is independent from **ZFC**.

§2 Large cardinals and some preliminary reduction of incompleteness

Definition. (Well, sort-of)

A large cardinal axiom is a statement that goes beyond **ZFC** in terms of consistency strength, and serves as a natural milestone as we look for stronger theories. Large cardinal axioms transcend each other in consistency strength.

Two examples:

1. *Con*(**ZFC**)
2. "There is an inaccessible cardinal κ (i.e. a regular strong limit cardinal)."

These are pretty weak as it goes...

§2 Large cardinals and some preliminary reduction of incompleteness

$I_0 \longrightarrow$ huge $\longrightarrow$ extendible $\longrightarrow$ supercompact

supercompact $\longrightarrow$ Woodin

Ramsey $\longleftarrow$ measurable $\longleftarrow$ strong $\longleftarrow$ Woodin

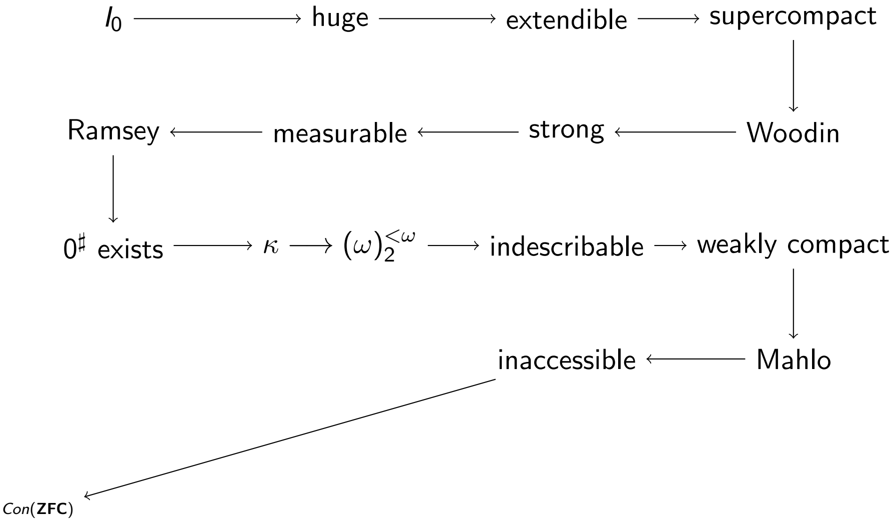
Ramsey $\longrightarrow$ $0^\sharp$ exists

$0^\sharp$ exists $\longrightarrow \kappa \longrightarrow (\omega)_2^{<\omega} \longrightarrow$ indescribable $\longrightarrow$ weakly compact

weakly compact $\longrightarrow$ Mahlo

inaccessible $\longleftarrow$ Mahlo

§2 Large cardinals and some preliminary reduction of incompleteness



§2 Large cardinals and some preliminary reduction of incompleteness

Definitions.

- An elementary embedding $j : \mathfrak{M} \longrightarrow \mathfrak{N}$ is a (one-to-one) mapping from one structure to another that respects first-order satisfaction (so $\mathfrak{M} \models \phi(x) \leftrightarrow \mathfrak{N} \models \phi(j(x))$, and similarly for sequences $\vec{x}$).
- j is non-trivial iff it isn't the identity map.
- In the context of set-theoretic structures, the least ordinal moved by j is known as the critical point of j and is denoted by '$crit(j)$'.

Things are subtle once we dig into the details, but we're going to look at the kinds of large cardinal we can obtain by examining elementary embeddings from a $j : V \longrightarrow \mathfrak{M}$, for transitive $\mathfrak{M}$.

§2 Large cardinals and some preliminary reduction of incompleteness

Definition.

A cardinal κ is measurable iff it is the critical point of a non-trivial $j : V \longrightarrow \mathfrak{M}$.

Already measurables are quite strong:

Theorem.

A measurable cardinal κ is an inaccessible limit of inaccessibles (in fact, there are κ-many inaccessibles below κ).

§2 Large cardinals and some preliminary reduction of incompleteness

Things get better:

Theorem.

[Scott] If there is a measurable cardinal, then $V \neq L$.

Also, note that because of the downward absoluteness of the definitions of inaccessible etc. these cannot refute $V = L$.

§2 Large cardinals and some preliminary reduction of incompleteness

Moreover, they look like natural principles:

> *"...the main reason to study them is the more open-minded interest in the properties which follow from assuming that very large cardinals exist; we want to consider the universe of set theory as being the cumulative type structure, continued through all possible ordinals, so that if it is possible to go so far that we get to a cardinal that is measurable, then we should do so." ([Drake, 1974], p186)*

Let's keep climbing..

Definition.

A cardinal κ is Woodin iff for all $A \subseteq V_\kappa$ there are arbitrarily large $\delta < \kappa$ such that for all $\lambda < \kappa$ there is an elementary embedding $j : V \longrightarrow \mathfrak{M}$ with critical point δ such that:

(i) $j(\delta) > \lambda$.

(ii) $V_\lambda \subseteq \mathfrak{M}$.

(iii) $A \cap V_\lambda = j(A) \cap V_\lambda$.

§3 More reduction and a limitation

We then have the following theorem:

Theorem.

[Martin and Steel, 1989] Suppose that there are ω-many Woodin cardinals. Then Projective Determinacy holds.

- I'm not going to prove this here, and with good reason.
- The proof is really, really, hard.
- Great! We've found a natural looking (insofar as anything in set theory is 'natural looking') class of axioms that can settle PD for us!
- But what about CH?

§3 More reduction and a limitation

Bad news:

THEOREM.

[Lévy and Solovay, 1967] Suppose that $\mathbf{T} = \mathbf{ZFC}+$ "There exists a measurable cardinal" is consistent. Then so is $\mathbf{T} + \neg$CH and $\mathbf{T} +$ CH.

i.e. No known large cardinals will help you solve CH.

§4 Extendability to inconsistency

- How far can we push the process?
- If you dig into characterisations of large cardinals using embeddings, you'll notice that the strength of the axioms depends on three crucial parameters:
  1. The similarity between V and $\mathfrak{M}$. (i.e. The more you close $\mathfrak{M}$ under the existence of things in V, the stronger the axiom).
  2. Where j sends the ordinals.
  3. How many j there are.

§4 Extendability to inconsistency

Compare and contrast:

- A cardinal κ is measurable iff κ is the critical point of a non-trivial e.e. $j : V \longrightarrow \mathfrak{M}$.
- A cardinal κ is:
  - γ-strong iff κ is the critical point of a non-trivial e.e. $j : V \longrightarrow \mathfrak{M}$ with $V_\gamma \subseteq \mathfrak{M}$.
  - strong iff it is γ-strong for all γ.
- A cardinal κ is Woodin iff for all $A \subseteq V_\kappa$ there are arbitrarily large $\delta < \kappa$ such that for all $\lambda < \kappa$ there is an elementary embedding $j : V \longrightarrow \mathfrak{M}$ with critical point δ such that:
  1. $j(\delta) > \lambda$.
  2. $V_\lambda \subseteq \mathfrak{M}$.
  3. $A \cap V_\lambda = j(A) \cap V_\lambda$.
- A cardinal κ is:
  - λ-supercompact iff there is a non-trivial e.e. $j : V \longrightarrow \mathfrak{M}$ such that $j(\kappa) > \lambda$ and ${}^\lambda\mathfrak{M} \subseteq \mathfrak{M}$.
  - A cardinal κ is supercompact iff it is λ-supercompact for all λ.

Well, let's start with the similarity between V and $\mathfrak{M}$ and shoot for as strong as we can get, and see what we end up with:

Definition.

A cardinal κ is Reinhardt iff κ is the critical point of a non-trivial $j : V \longrightarrow V$.

Unfortunately, we'll now show the following:

Theorem.

[Kunen, 1971] Assuming **ZFC**, there are no Reinhardt cardinals.

§4 Extendability to inconsistency

We'll prove this assuming the following:

Lemma.

(Solovay Splitting Lemma) Let κ be a regular uncountable cardinal, and let $S \subseteq \kappa$ be stationary. Then S may be written as a disjoint union of κ-many stationary sets (i.e. There is a sequence $\langle S_\zeta | \zeta < \kappa \rangle$ such that $S_\zeta \subseteq S$ is stationary in κ for every $\zeta < \kappa$, $S_\zeta \cap S_{\zeta'}$ for all $\zeta, \zeta' < \kappa$, and $S = \bigcup_{\zeta<\kappa} S_\zeta$).

- So, we've seen that, in a sense, large cardinals defined by embeddings represent natural looking principles.
- However, the template can be extended to inconsistent principles.
- But are they really inconsistent simpliciter?

§5 Some philosophical remarks

- The normal proof of the Solovay Splitting Lemma required the use of AC.
- In fact all known proofs of the non-existence of Reinhardt cardinals require Choice.
- What if Choice were false? Could we keep going?

Question.

Are there any embedding principles that are inconsistent with **ZF**?

§5 Some philosophical remarks

- Peter Koellner and Hugh Woodin have been examining choiceless large cardinals, searching for a deep inconsistency.

Definition.

- A cardinal κ is γ-Reinhardt iff it is the critical point of a non-trivial $j : V \longrightarrow V$ with $j(\kappa) > \gamma$.
- A cardinal κ is Super Reinhardt iff it γ-Reinhardt for all γ.

Definition.

A cardinal κ is Berkeley iff for every transitive set M containing κ, and every ordinal $\delta < \kappa$, there is a non-trivial e.e. $j : M \longrightarrow M$ with critical point between δ and κ. That's a whole lot of embeddings!

- Thus far, no inconsistency has emerged, and in investigating the 'properties' of these 'cardinals' there have been some elegant consistency implications discovered.

- The existence of these implications and lack of an immediate contradiction is somewhat unnerving.
- This is especially so if we think that the job of set theory is to landmark consistency strength.
- Should we view Choice as a limitative principle in some sense (in analogy with $V = L$ and measurables)?

§5 Some philosophical remarks

Moreover, CH is still left untouched (not to mention the violence done to our theory of cardinals!). We now have two challenges before us:

Challenge I.

Come up with a kind of principle that motivates the large cardinals required for PD on independent grounds (even better if the idea avoids extendability to inconsistency in a motivated fashion).

Challenge II.

Come up with a kind of principle that can settle CH.

I'll leave you with...

"On the other hand, from an axiom in some sense opposite to this one [$V = L$], the negation of Cantor's conjecture could perhaps be derived. I am thinking of an axiom which (similar to Hilbert's completeness axiom in geometry) would state some maximum property of the system of all sets, whereas axiom A [i.e. $V = L$] states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set..." ([Gödel, 1964], p262-263, footnote 23)

Thanks! Discussion!

Cohen, P. (1963).
The independence of the continuum hypothesis.
Proceedings of the National Academy of Sciences of the United States of America, 50:1143–1148.

Drake, F. R. (1974).
Set Theory: An Introduction to Large Cardinals.
North Holland Publishing Co.

Gödel, K. (1940).
The Consistency of The Continuum Hypothesis.
Princeton University Press.

Gödel, K. (1964).
What is Cantor's continuum problem?
In *[Gödel, 1990]*, pages 254–270. Oxford University Press.

Gödel, K. (1990).
Collected Works, Volume II: Publications 1938-1974.
Oxford University Press.
Edited by: Solomon Feferman (Editor-in-chief), John W. Dawson, Jr., Stephen C. Kleene, Gregory H. Moore, Robert M. Solovay, Jean van Heijenoort.

Kunen, K. (1971).
Elementary embeddings and infinitary combinatorics.
The Journal of Symbolic Logic, 36:407–413.

Lévy, A. and Solovay, R. (1967).
Measurable cardinals and the continuum hypothesis.
Israel Journal of Mathematics, 5(4):234–248.

Martin, D. A. and Steel, J. R. (1989).
A proof of projective determinacy.
Journal of the American Mathematical Society, 2(1):71–125.