# Potentialist Sets, Intensions, and Non-Classicality

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## Introduction and motivation

-A view that's been popular in the philosophy of set theory recently:

**Set-Theoretic Potentialism.** Set-theoretic potentialism is the view in the philosophy of mathematics that the universe of set theory is never fully completed, but rather unfolds gradually as parts of it increasingly come into existence or become accessible to us ([Hamkins and Linnebo, 2018], p. 1)

-This view divides into strict and liberal versions:

-*Liberal potentialists* regard the modal truths as unproblematic. In particular, there are modal truths about generative processes in their entirety, including those that cannot be completed...

-*strict potentialism*...goes beyond the liberal view by requiring, not only that every object be generated at some stage of a process, but also that every truth be "made true" at some stage. ([Linnebo and Shapiro, 2019], p. 169)

-It is natural to formalise potentialism in a modal framework (which we'll discuss below).

-The situation is complicated by the issue of what *classes* are like for the potentialist.

-One nice answer:

**The Intensional Account of Classes.** Sets are *extensional entities* in that it is sufficient for two sets to be identical that they have the same members, and they have the same members in all possible worlds. Classes are *intensional entities*—they can *change members* across different possible worlds and just because two classes have the same members over a domain does not make them the same class.

-If we have a non-potentialist outlook, the intensional account doesn't seem to buy us any difference between sets and classes.

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-For a potentialist, however, we *can* think of classes as intensional entities.

-But how should we handle talk of sets and intensions for the potentialist?

**Main Aims.**

–(1.) Introduce some of the existing work here.

–(2.) Provide a semantics for potentialist sets and intensions. (**Note:** I'll skip this.)

–(3.) Argue that the kind of potentialism in play affects the logic you get out (in particular by how much it extends intuitionistic logic). (**Note:** I'll skip this.)

–(4.) Address some philosophical issues concerning potentialist classes.

Here's the plan:

-§1 Potentialist Systems, Sets, and Intensions

-§2 Potentialist Semantics (**Note:** Skip this unless you're keen!)

-§3 Relationship to category theory (**Note:** Definitely skip this unless you're ultra-keen!)

-§4 Discussion

—§4.1 Is all this anti-potentialist?

—§4.2 What intensions?

—§4.3 Motivating strict potentialism

-§5 Open questions (and conclusions)

# 1 Potentialist Systems, Sets, and Intensions

-There are various kinds of potentialism we can consider.

-We might think of the sets as being brought into existence via the introduction of powersets, or by adding forcing generics, and so on...

**Definition 1.** [Hamkins and Linnebo, 2018] A *potentialist system* (for sets) is a pair $\mathbb{S} = (S, \leqslant_{\mathbb{S}})$, where $S$ is a collection of structures of a certain kind and $\leqslant_{\mathbb{S}}$ is a refinement of the substructure relation.

-For metamathematical ease we will fix some initial countable transitive model $M \models \mathsf{ZFC}$.

**Definition 2.** *The Rank Potentialist System* is the following potentialist system:

(i) $S = \{V^M_\beta | \beta \in On^M\}$

(ii) $V^M_\alpha \leqslant_{\mathbb{S}} V^M_\beta$ iff $\alpha \leqslant \beta$.

**Definition 3.** *The Generic Multiverse Potentialist System* is the following potentialist system:

(i) $S = \{M[G] | \text{"}G \text{ is } M\text{-generic"}\}$

(ii) $M[G] \leqslant_{\mathbb{S}} M[H]$ iff $M[H]$ is a forcing extension of $M[G]$.

**Ignore this definition. It is relevant but fiddly.**

**Definition 4.** *The Steel Multiverse Potentialist System* is the following potentialist system:

(i) $S = \{M[G \restriction \alpha] | \text{"}G \text{ is } Col(\omega, < Ord^M)\text{-generic and } \alpha \in Ord^{M}\text{"} \}$

(ii) $M[G] \leqslant_{\mathbb{S}} M[H]$ iff $M[H]$ is a forcing extension of $M[G]$.

**Definition 5.** *The Countable Transitive Potentialist System* is the following potentialist system:

(i) $S = \{N | \text{ "}N \text{ is a countable transitive model of ZFC containing } M\text{"}\}$

(ii) $N \leqslant_{\mathbb{S}} N'$ iff $N \subseteq N'$

**Definition 6.** *The Actualist (Potentialist) System* is the following potentialist system:

(i) $S = \{M\}$

(ii) $M \leqslant_{\mathbb{S}} M$ iff $M = M$

-We now want to add in intensions.

-These can be thought of as functions from $S$ to $\bigcup_{W \in S} W$ (note that since everything I'm considering is transitive, $\bigcup_{W \in S} W$ is well defined).

-For the sake of ease, I want to consider only those intensions which have ranges *contained* in the relevant worlds:

**Definition 7.** A function $f : S \to \bigcup_{W \in S} W$ is $\mathbb{S}$-*bounded* iff $f(W) \subseteq W$ for every $W \in S$.

**Definition 8.** An *intensional* potentialist system is a triple $\mathbb{S} = (S, \leqslant_{\mathbb{S}}, \mathcal{X})$, where $\mathcal{X}$ is a collection of $\mathbb{S}$-bounded functions.

-Two further definitions are going to be of use later:

**Definition 9.** A class $f$ is *monotone* or *montonic* iff $x \in f(W)$ implies that $x \in f(W')$ for all $W' \geqslant_{\mathbb{S}} W$.

-Or a stronger condition:

**Definition 10.** A class $f$ is *stable* iff:

(i) it is monotone, and

(ii) $x \notin f(W)$ and $x \in W$ implies that $x \notin f(W')$ for all $W' \geqslant_{\mathbb{S}} W$.

## 2 Potentialist Semantics

**Ignore this section unless you want to think about how to approach the semantics for intensional classes mathematically.** I summarise what you need to know at the start of §4. It's also **not** fully mathematically precise yet.

-We have the *modalised* language of set theory $\mathscr{L}_{\in}^{\Diamond}$, that is just $\mathscr{L}_{\in}$ except with the modal operators $\Diamond$ and $\Box$ added.

**Definition 11.** We define $(S, \leqslant_{\mathbb{S}})$ *satisfying* $\phi$ *at* $W$ *under* $v$ via the following recursive clauses:

(i) $W \Vdash_v x \in y$ iff $v(x) \in v(y)$

(ii) $W \Vdash_v \neg\phi$ iff $W \nVdash_v \phi$

(iii) $W \Vdash_v \phi \vee \psi$ iff $W \Vdash_v \phi$ or $W \Vdash_v \psi$

(iv) $W \Vdash_v \exists x\phi$ iff $W \Vdash_{v_1} \phi$ for some valuation $v_1$ that is like $v$ except possibly on $x$, but where $v_1(x)$ is a member of $W$

(v) $W \Vdash_v \forall x\phi$ iff $W \Vdash_{v_1} \phi$ for every valuation $v_1$ that is like $v$ except possibly on $x$, but where $v_1(x)$ is a member of $W$

(vi) $W \Vdash_v \Box\phi$ iff every $W' \geqslant_{\mathbb{S}} W$ is such that $W' \Vdash_v \phi$

(vii) $W \Vdash_v \Diamond\phi$ iff some $W' \geqslant_{\mathbb{S}} W$ is such that $W' \Vdash_v \phi$

-We have the usual *non-modal* language of set theory $\mathscr{L}_{\in}$, that has variables for sets and a single non-logical symbol $\in$.

**Definition 12.** A potentialist system $\mathbb{S}$ of $\mathscr{L}$-structures provides a potentialist account of a particular $\mathscr{L}$-structure $N$, if every world $W \in S$ is a substructure of $N$ and furthermore, for every such $W$ and every individual $a \in N$, there is a world $U \in S$ accessed by $W$ with $a \in U$. ([Hamkins and Linnebo, 2018])

-We can now think of translating between our modal and non-modal theory as follows. To go from $\mathscr{L}_{\in}$ to $\mathscr{L}_{\in}^{\Diamond}$, replace every occurrence of $\exists$ with $\Diamond\exists$ and each occurrence of $\forall$ with $\Box\forall$ (denote the translation of $\phi$ by $\phi^{\Diamond}$).

**Theorem 13.** *[Linnebo, 2013], [Hamkins and Linnebo, 2018] (Model-Theoretic Mirroring) If $\mathbb{S}$ provides a potentialist account of a structure $N$, then truth in $N$ is equivalent to potentialist truth at the worlds of $W$. Namely, for any $\mathscr{L}$-formula $\psi$ and any $a_0$,..,$a_n \in N$, we have*

$$N \models \psi(a_0, ..., a_n) \leftrightarrow W \models \psi^{\Diamond}(a_0, .., a_n)$$

*for any $W \in S$ in which the individuals $a_0, ..., a_n$ exist.*

-We have the *non-modal* language of set theory with *intensions* $\mathscr{L}_{\in,\eta}$.

—We add variables $C_0$, ..., $C_n$, ... for intensions.

—We add a non-logical predicate $\eta$ to hold between set-variables and intension variables.

-We then have the *modal* language of of set theory with *intensions* $\mathscr{L}^{\Diamond}_{\in,\eta}$.

-We then define satisfaction on a valuation as before:

**Definition 14.** We define $(S, \leqslant_{\mathbb{S}}, \mathcal{X})$ *satisfying* $\phi$ *at* $W$ *under* $v$ via the following recursive clauses:

(i) $W \Vdash_v x\eta Y$ iff $v(x) \in v(Y)(W)$

(ii) All the previous clauses for $\mathscr{L}^{\Diamond}_{\in}$ and expected compositional clauses.

(iii) $W \Vdash_v \exists X\phi$ iff $W \Vdash_{v_1} \phi$ for some valuation $v_1$ that is like $v$ except possibly on $X$

(iv) $W \Vdash_v \forall X\phi$ iff $W \Vdash_{v_1} \phi$ for every valuation $v_1$ that is like $v$ except possibly on $X$

**Conjecture.** I *strongly* expect that a mirroring theorem will be available for two-sorted class-theories. So where $(N, \mathcal{C})$ is a model of a theory in $\mathscr{L}_{\in,\eta}$ (e.g. NBG or NBG-Powerset) we can come up with potentialist accounts of truth in $(N, \in, \eta, \mathcal{C})$ via some $\mathbb{S} = (S, \leqslant_{\mathbb{S}}, \mathcal{X})$.

**Idea.** Restrict to stable intensions, and just chunk up any $C \in \mathcal{C}$ into $C \cap W$ at every $W \in \mathbb{S}$. We then say that $x\eta C$ (in $\mathscr{L}_{\in,\eta}$) iff there is a $W$ such that $\Diamond x\eta C(W)$. Call this the **bivalent** interpretation.

**Conjecture.** Mirroring Theorems also come in syntactic flavours. I strongly suspect there is mirroring there too, but it's more difficult to show.

-This mirroring is good as far as it goes, but it's a bit annoying in certain respects.

-Whether or not a set is in a class becomes *too* coarse.

-Here's some example intensions (or at least conditions that will give you a $\mathbb{S}$-bounded function in the obvious way).

-e.g.1. $x = x$.

-e.g.2. $x$ is constructible $Constructible(x)$

-e.g.3. $x$ is countable $Countable(x)$

-$x = x$ is clearly stable.

-But $Constructible(x)$ (under rank potentialism) and $Countable(x)$ (under forcing potentialisms) are not.

-Instead of classical modal logic, let's adopt a free logic (something along these lines has already been done by Brauer-Linnebo-Shapiro).

-Let's consider the languages obtained when we just add in constants for every possible set and/or intension. I'll denote these by a bar over the script-L (e.g. $\mathscr{L}^{\Diamond}_{\in}$).

-Assume the potentialist translation of $V = L$ under rank potentialism, and consider whether a set is constructible vs. whether it's self-identical.

-Consider whether a particular set $x$ is countable under forcing potentialisms.

**Slogan form of project.** Ask not *if* a set $x$ gets in to some class $C$, but **when**.

**Definition 15.** The *Boolean* intensional semantics is given by

$$[\![\phi]\!]_{\mathbb{S}} = \{W \in S | W \models \phi\}$$

-**Note:** I am fairly confident (though haven't checked the details) that this will validate classical logic.

-**Note:** You can always quotient down the Boolean interpretation into a bivalent one just by putting every non-empty $[\![\phi]\!]_{\mathbb{S}}$ onto $1$ in the usual two-element Boolean algebra.

-This is all fine for the *liberal* potentialist modelling their attitude to truth, they think they have access to the whole modal process.

-But what about the *strict* potentialist?

-They think that things are *made* true.

-Instead of a Boolean approach to, it's much more natural to take a Kripke style semantics:

**Definition 16.** The *Kripke-style* semantics is given as follows:
$[\![\phi]\!]_{\mathbb{S}} = \{W \in S | \forall W' \geqslant_{\mathbb{S}} W, W' \models \phi\}$

-i.e. The truth-values correspond to "times in $\mathbb{S}$ from which $\phi$ is forevermore true".

-These will give us more fine-grained truth values that help us account for e.g. a particular set taking a long time to get into an intension.

-**Problem:** Get traction on these truth-values in determining properties of the logic.

# 3 Relationship to category theory

**Definitely ignore this section unless you are interested in the category-theoretic side of things.** *Summary*: There's some nice relationships with category theory, the semantics I talked about, and monotonic classes.

**Definition 17.** A *functor* $F : \mathscr{C} \to \mathscr{D}$ between categories $\mathscr{C}$ and $\mathscr{D}$ is a map that:

(i) Assigns every $\mathscr{C}$-object $c$ a $\mathscr{D}$-object $F(c)$.

(ii) Assigns every $\mathscr{C}$-arrow $f : c_0 \to c_1$ a $\mathscr{D}$-arrow $F(f) : F(c_0) \to F(c_1)$ such that:

  (a) For every $\mathscr{C}$-object $c$, $F(Id_c) = Id_{F(c)}$, and

  (b) For every pair of $\mathscr{C}$-arrows $f : c_0 \to c_1$ and $g : c_1 \to c_2$, $F(g \circ f) = F(g) \circ F(f)$.

**Definition 18.** A *natural transformation* between functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{C} \to \mathscr{D}$ is a family of $\mathscr{D}$-arrows $\tau_c : F(c) \to G(c)$, one for each $\mathscr{C}$-object $c$, such that all squares of the following form commute (for all $\mathscr{C}$-objects $a, b$ and all $\mathscr{C}$-arrows $f : a \to b$):

$$\begin{array}{ccc} F(a) & \xrightarrow{\tau_a} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow{\tau_b} & G(b) \end{array}$$

**Definition 19.** The *category of functors between $\mathscr{C}$ and $\mathscr{D}$* (often written $\mathscr{D}^{\mathscr{C}}$) is the category that has as objects all functors from $\mathscr{C}$ to $\mathscr{D}$ and as arrows all natural transformations between functors.

**Fact 20.** *If $\mathscr{C}$ is a small (i.e. set-sized) category, then $\mathbf{Set}^{\mathscr{C}}$ forms a topos.*

-In particular $\mathbb{S}$ is just such a category!

-Monotonic $\mathbb{S}$-bounded functions can be thought of as represented by (some) functors in $\mathbf{Set}^{\mathbb{S}}$.

-In particular, such a functor $F$ will be a subobject of the following functor in $\mathbf{Set}^{\mathbb{S}}$:

**Definition 21.** $dom\text{-}S : S \to \mathbf{Set}$ is defined as the following functor:

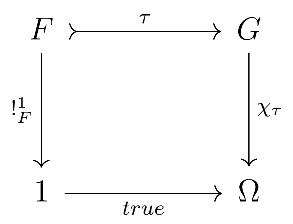
(i) For every $W \in S$, $dom\text{-}S(W) = \bigcup S$

(ii) For every $f : W \to W'$ in $\mathbb{S}$, $dom\text{-}S(f) = Id_{\bigcup S}$

-Topoi come equipped with the following gadget:

**Definition 22.** If $\mathscr{C}$ is a category with a terminal object $1$, then a *subobject classifier* for $\mathscr{C}$ is a $\mathscr{C}$-object $\Omega$ and a $\mathscr{C}$-arrow $true : 1 \to \Omega$ (sometimes written $\top : 1 \to \Omega$) with:

**($\Omega$-Axiom)** For every monic $\tau : F \rightarrowtail G$ there is exactly one $\mathscr{C}$-arrow $\chi_\tau$ (the characteristic arrow of $\tau$) such that the following diagram is a pullback square:



The character $\chi_\tau : G \to \Omega$ of $\tau$ is defined at each $W$, and has components $(\chi_\tau)_W : G(W) \to \Omega(W)$, which in turn are defined coordinate-wise:

$$\text{for each } x \in G(W), (\chi_\tau)_W(x) = \{f : W \to W' | G(f)(x) \in F(W')\}$$

$\chi_\tau(x)$ can be thought of as denoting the first time $N$ (according to $\mathbb{S}$) at which $x$ is determined to be in the subobject $F$.

**Definition 23.** For $x \in \bigcup S$ (i.e. the whole domain of our potentialist system) and monotonic intensional class $f$ determining a functor $F : \mathbb{S} \to \mathbf{Set}$ we assign the sentence $x \in F$ (thought of a propositional variable) to the truth value $Val_{x\eta F} : 1 \to \Omega$ in $\mathbf{Set}^{\mathbb{S}}$ that picks, at a world $W$ the following element of $\Omega(W)$:

$$\{f : W \to W' | x \in f(W')\}$$

i.e. $Val_{f \in x}(W) : 1(W) \to \Omega(W)$ is defined component-wise by mapping the unique element of $1(W)$ to the set of those (arrows to) worlds $W'$ for which $x \in f(W')$.

**Definition 24.** Let $\perp$ be the character of $!^0_1 : 0 \to 1$. Then $\neg : \Omega \to \Omega$ is the character of $\perp$ in $\mathscr{E}$, i.e. $\neg = \chi_\perp$.

**Definition 25.** $\wedge : \Omega \times \Omega \to \Omega$ is the character in $\mathscr{E}$ of the product arrow $\langle \top, \top \rangle : 1 \to \Omega$.

**Definition 26.** $\vee : \Omega \times \Omega \to \Omega$ is the character of the image of the arrow $[\langle \top_\Omega, Id_\Omega \rangle, \langle Id_\Omega, \top_\Omega \rangle] : \Omega + \Omega \to \Omega \times \Omega$

**Definition 27.** Let $e : \leqslant \rightarrowtail \Omega \times \Omega$ be the equaliser of $\wedge : \Omega \times \Omega \to \Omega$ and $pr_1 : \Omega \times \Omega \to \Omega$ (where $pr_1$ is the projection onto the first coordinate of $\Omega \times \Omega$). Then $\to : \Omega \times \Omega \to \Omega$ is the character of $e$ (i.e. $\to = \chi_e$).

**Definition 28.** Given a topos $\mathscr{E}$, let $\mathscr{E}(1, \Omega)$ denote the collection of all arrows from the terminal object to $\Omega$. A $\mathscr{E}$-valuation is a function $Val$ that assigns each propositional variable $P$ a truth value $f : 1 \to \Omega$. $Val$ in turn determines a value for each propositional well-formed formula $\phi, \psi, ...$ etc. recursively via the following compositional clauses:

(i) $Val(\neg \phi) = \neg \circ Val(\phi)$

(ii) $Val(\phi \wedge \psi) = \wedge \circ \langle Val(\phi), Val(\psi) \rangle$

(iii) $Val(\phi \vee \psi) = \vee \circ \langle Val(\phi), Val(\psi)\rangle$

(iv) $Val(\phi \to \psi) = \to \circ \langle Val(\phi), Val(\psi)\rangle$

Furthermore, we say that $\phi$ is $\mathscr{E}$-valid (or $\mathscr{E} \models \phi$) iff every $\mathscr{E}$ valuation $Val$, $Val(\phi) = \top : 1 \to \Omega$.

-This coincides with the propositional variables and connectives with the account given earlier.

-We already knew that Kripke-semantics over some $\mathbb{P}$ and valuations from $\mathbf{Set}^{\mathbb{P}}$ coincide.

-This gives us traction at least on the propositional case.

**Theorem 29.** *[Dummett, 1959], [Segerberg, 1968] (The Dummett-Segerberg Theorem) Suppose that $\mathbb{S}$ is an infinite linearly ordered poset. Then $\mathbf{Set}^{\mathbb{S}} \models \phi$ iff $\phi$ is a theorem of intuitionistic propositional logic with all classical tautologies of the form $(\psi_0 \to \psi_1) \vee (\psi_1 \to \psi_0)$ added (so called Dummett's Logic).*

**Fact 30.** *For partial order $\mathbb{P}$, the topos $\mathbf{Set}^{\mathbb{P}}$ satisfies the weak excluded middle law $\neg\phi \vee \neg\neg\phi$ if $\mathbb{P}$ is confluent, i. e. for all $x \leqslant_{\mathbb{P}} y, y'$, there is a $z$ with $y, y' \leqslant_{\mathbb{P}} z$.*

-It's interesting that the link is **so close** between the objects of $\mathbf{Set}^{\mathbb{S}}$ and intensional classes.

-**Note:** It's not completely clear to me how to get traction on this in the first-order case.

# 4 Discussion

**Here's a summary of §2 and §3:**

1. We start with some countable transitive model $M$.
2. We use $M$ to give a description of a potentialist system $\mathbb{S} = (S, \leqslant_{\mathbb{S}}, \mathcal{X})$, where $\mathcal{X}$ is some collection of intensional classes for $\mathbb{S}$.
3. We have the languages $\mathscr{L}_{\in}$ (non-modal set theory), $\mathscr{L}_{\in}^{\Diamond}$ (modal set theory), $\mathscr{L}_{\in,\eta}$ (non-modal class theory), $\mathscr{L}_{\in,\eta}^{\Diamond}$ (modal class theory).
4. Roughly as in the usual case, we can use the possible worlds to give a semantics for claims in $\mathscr{L}_{\in,\eta}^{\Diamond}$.
5. But we also want a semantics for *non-modal* claims.
6. For $\mathscr{L}_{\in}$ under **liberal** potentialism we have the mirroring theorems for rank potentialism [Linnebo, 2013] and (something closely related to) countable transitive model

potentialism [Scambler, 2020]. (More general versions of mirroring are considered in [Hamkins and Linnebo, 2018].)

**Side-Quest(ion).** What's the situation for *generic multiverse potentialism*?

7. For **liberal** potentialism we can **probably** get similar(ish) kinds of mirroring between $\mathscr{L}^{\Diamond}_{\in,\eta}$ and $\mathscr{L}_{\in,\eta}$.

8. But what about **strict** potentialism?

9. We can then use $(\mathbb{S}, \in, \mathcal{X})$ to give a semantics for **non-modal** claims about intensional classes.

10. One way is to use the **flat** interpretation: $x\eta F$ is true iff $M \models \Diamond x\eta F$.

11. This kind of mirroring is fine as far as it goes: But it **semantically blurs** important distinctions.

12. e.g. Consider the following intensions.

    - $x = x$ (I'll abbreviate this $\mathsf{Ident}(x)$)

    - $x$ is countable, $\mathsf{Count}(x)$

    - $x$ is constructible, $\mathsf{Const}(x)$.

13. These have **differing modal profiles** on each version of potentialism, and you may have to **wait longer** and membership of a class in a set is **not guaranteed** (even if **possible**)!

14. **Proposed solution:** (Still to be worked out in detail.) Adopt a Kripke-Beth-Joyal semantics on which (roughly speaking) $\phi$ gets the value (a $\mathbb{S}$-hereditary subset of $S$) according to when $\phi$ is true **now and forevermore**.

15. The **space of truth values** is basically a Heyting algebra that you get from considering $\mathbb{S}$-hereditary sets (subsets of $\mathbb{S}$ that are upwards closed).

    -(**Note:** This has (basically, with minimal tweaking) been done independently by Brauer-Shapiro-Linnebo for $\mathscr{L}_{\in}$ and $\mathscr{L}^{\Diamond}_{\in}$.)

    -**However**, how to handle **intensions** is still open.

16. **My idea:** $x\eta F$ gets the value for when $x$ **gets into** $F$ (so long as $F$ is monotonic).

17. There are some interesting links to category theory (§3) but let's not worry about them here.

-So, that's the **rough** idea.

-I want to consider some **philosophically relevant** issues.

## 4.1 Is all this anti-potentialist?

-Here's one kind of **objection** I get a lot.

-You ascribe **unbounded** collections of worlds as truth values!

-Isn't this all just **anti**-potentialist and *worthless* from their perspective?

-I find this response very **unpersuasive**.

-We are looking at a **model theory** for potentialism.

-Yes the model may be imperfect, but it serves as a guide to what we think is **true**

-Consider the following contrast case.

-You are a friendly neighbour **actualist**.

-Truth in $\mathscr{L}_\in$ in $V$ isn't definable for you (without classes).

-But what about saying the following:

-We assume that there is a countable transitive model wee-$V$ elementarily equivalent to $V$.

-Introduce a constant for wee-$V$.

-I now reason about wee-$V$, I can talk about what I can force over wee-$V$ and more besides.

-I then look at what first-order sentences get what truth values (either $0$ or $1$) in wee-$V$.

-I then export these truth values back to $V$ (cf. [Barton, 2020], [Antos et al., 2021]).

-But I did nothing wrong by forcing over wee-$V$, considering height extensions of wee-$V$ etc.

-Why not so for the potentialist?

-I think the universe is potential, it's some $\mathbb{S}$ that I can't get a grip on.

-To model this I look at wee-$\mathbb{S}$ built over $M$.

-By analysing how I can ascribe truth values there, I come to realise that because I'm a strict potentialist I should accept some logic or other (e.g. what validities of classical logic fail).

-What's wrong with this?

-One **disanalogy**: The truth values **themselves** are proper classes here, rather than it simply being the case that we've got $0$ and $1$ and its just undefinable which gets assigned to what sentence.

-I think this is still **OK**: Let the truth values be the wee-$\mathbb{S}$-hereditary sets for all it matters for understanding the logic of the system (even if the model isn't **perfect**).

## 4.2 What intensions?

-In the semantics I give, I just assumed that we're given some **fixed stock** of intensions from the **outset**.

-This is largely just for **ease**—I'm trying to get traction on the problem and don't want to complicate things more than needed.

-There's **several** questions we might ask about this.

**Question.** Constant or variable domain of intensions?

-Often we think of intensions as provided by **application conditions**.

-If you think that these application conditions must be somehow **definable**, then it's very plausible that you should adopt a **variable** domain of intensions.

-As **more** sets come into being, we are able to define **more** intensions using them.

-(**Note:** I think it's not too problematic to incorporate variable domain into the semantics given earlier.)

-Perhaps this relates to the **free** vs. **classical** modal logic perspectives?

-Do you have the **names** for the set (and ability to define classes in terms of them) **before** you have the set?

-Maybe there is something like the **strict** or **liberal** potentialist divide going on here?

**Question.** How much comprehension?

-Questions about comprehension don't make **too** much sense beyond what the various $(W, \mathcal{X}^W)$ satisfy.

-The rank potentialist thinks (roughly speaking) that (non-modally) $V \models$ ZFC and the countable transitive model potentialist thinks that $V \models$ ZFC$^-$ + "Every set is countable".

-But there are going to be some limitations here especially with the free logic perspective under generic multiverse potentialism.

-Let $G$ and $H$ be incompossible. Then the instance of comprehension given by $\phi(x) = x \in G \vee x \in H$ is always going to fail (i.e. the set:

$$\{x | x \in G \vee x \in H\}$$

is not a class at any world.

-Perhaps this relates to the fact that a **limit structure** for the generic multiverse potentialist is not going to have a sensible theory (e.g. Replacement will fail).

-For the other views, there should be at least **some** sensible accounts of how much comprehension can satisfied. (e.g. Given some $V \models \mathbf{T}$, just look at $(V, Def(V))$ (this will model your NBG-style theory) and then chunk up each member of $Def(V)$ into its stable/monotonic parts.)

-Can we get beyond **definable** classes (in the limit)?

-Perhaps truth predicates (for the **liberal** potentialist)?

-(**Note:** Things are actually oddly complex for the forcing potentialist.)

**Question.** What modal profiles should we allow/incorporate?

-Earlier we made the following definitions:

**Definition 31.** A class $f$ is *monotone* or *montonic* iff $x \in f(W)$ implies that $x \in f(W')$ for all $W' \geqslant_{\mathbb{S}} W$.

-And the stronger condition:

**Definition 32.** A class $f$ is *stable* iff:

(i) it is monotone, and

(ii) $x \notin f(W)$ and $x \in W$ implies that $x \notin f(W')$ for all $W' \geqslant_{\mathbb{S}} W$.

-I only know how to handle **monotonic** classes.

-Some I think are clearly out, e.g.

**Definition 33.** An intensional class $f$ is *capricious* iff for every set $x \in \bigcup S$ and every world $W$ with $x \in W$, there are $W_1, W_2 \geqslant_{\mathbb{S}} W$ such that $x \notin f(W_1)$ and $x \in f(W_2)$.

-There's nothing reasonable (beyond **modal** stuff) to say about membership in capricious classes. They constantly change their mind.

-(That's not to say they're not legitimate, e.g. work under rank potentailism and let $f(V_\alpha) = \emptyset$ at all successor stages and $f(V_\lambda) = \omega$ at every limit stage.)

-But others are interesting, e.g.

**Definition 34.** An intensional class $f$ is *anti-monotone* or *anti-monotonic* iff for every $W$, $x \in W$ and $x \notin f(W)$ implies that $x \notin f(W')$ for all $W' \geqslant_{\mathbb{S}} W$.

-Plausibly we should be able to give truth-values to $x \not\eta F$ for an anti-monotone class.

-Anti-monotone classes are actually pretty thin on the ground (e.g. the $M$-uncountable sets under forcing potentialism).

-Similarly we can consider:

**Definition 35.** A class $f$ is *eventually convergent* iff there are worlds $W_1 \leqslant_{\mathbb{S}} W_2 \leqslant_{\mathbb{S}} W_3$ with $x \in f(W_2)$ taking the opposite (classical) value from $x \in f(W_1)$ and $x \in f(W_3)$, but where there is a world $W_4 \geqslant_{\mathbb{S}} W_3$ such that for all $W_5 \geqslant_{\mathbb{S}} W_4$, $x \in f(W_5)$ or $x \notin (W_5)$ uniformly (i.e. one of $x \in f$ or $x \notin f$ is true at every world past some point for some worlds).

-Some **important** classes are eventually convergent.

-For example the **uncountable sets** are eventually convergent under generic multiverse potentialism.

-More generally, we often think class theory should be closed under **sensible operations**.

**Definition 36.** Given an intensional class $F$, we let the *component-wise complement of* $F$ be the intensional class $F'$ given by the rule $F' = W - F(W)$.

-e.g. The **uncountable sets** denotes the component-wise complement of the countable sets.

-But the uncountable sets (under forcing potentialism), though **eventually convergent**, are **neither** stable nor monotonic.

-**Stable** classes *can* be complemented (the component-wise complement of a stable class is also stable).

-The complement of a monotonic class $F$, under strict potentialism, could also be thought of as the **largest montonic class disjoint** from $F$ everywhere.

-Maybe strict potentialism is more, well **strict**, about what modal profiles should count as legitimate (e.g. only monotonic classes are allowed).

## 4.3 Motivating strict potentialism

-One issue here concerns **how to motivate** strict potentialism.

-Especially in **this context** (we're playing the set theory game, and strict potentialism isn't as natural as in the context of, say free-choice sequences).

-This may be a **more general** worry than intensional classes.

-I see two key motivations:

(1.) **Procedural postulationism**.

-We obtain new sets by postulation ([Fine, 2005]).

-You can think of this as asking a genie for a (consistent) object of a particular kind.

-But genies are **tricksters**! (Thanks to Kam Williams for this way of putting it.)

-So maybe they give you **nasty** objects (e.g. one half of a bad generic).

-This will put **constraints** on how the universe can develop further.

-You don't **know** what you're going to get until you execute the command.

-(Maybe you think commands should be more precise, but this needs to be argued, and in any case it's not reasonable to ask for this for any kind of **forcing potentialism**.)

(2.) **Knowledge/conceptual states**. The original motivation given in [Lear, 1977].

-We move to new possible worlds by **specifying them** and **expanding** our concept.

-Until we've developed a concept of those new worlds, they aren't **visible** to us.

# 5 Open questions (and conclusions)

Obviously it's early days, but I wanted to close with the **main points** and some directions for **going forward**:

(1.) I think there's real space for a **mirroring type phenomenon** to be explored here.

(2.) The semantics needs to be made **mathematically precise**.

(3.) But we should get **classical logic** for the **liberal potentialist** and **intuitionistic logic** for the strict potentialist.

(4.) What kind of intuitionistic logic you get depends on the **structure of** $\mathbb{S}$.

(5.) There are some **serious** philosophical questions to be asked about the **kind of intensions** we want to allow (especially for the strict potentialist).

**Question.** How to handle non-transitive contexts?

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