Potentialist Sets, Intensions, and Non-Classicality

Neil Barton[*]

18. and 24. November 2021

Introduction and motivation

-A view that's been popular in the philosophy of set theory recently:

Set-Theoretic Potentialism. Set-theoretic potentialism is the view in the philosophy of mathematics that the universe of set theory is never fully completed, but rather unfolds gradually as parts of it increasingly come into existence or become accessible to us ([Hamkins and Linnebo, 2018], p. 1)

-This view divides into strict and liberal versions:

-*Liberal potentialists* regard the modal truths as unproblematic. In particular, there are modal truths about generative processes in their entirety, including those that cannot be completed.

-*strict potentialism*...goes beyond the liberal view by requiring, not only that every object be generated at some stage of a process, but also that every truth be "made true" at some stage. ([Linnebo and Shapiro, 2019], p. 169)

-It is natural to formalise potentialism in a modal framework (which we'll discuss below).

-The situation is complicated by the issue of what *classes* are like for the potentialist.

-One nice answer:

The Intensional Account of Classes. Sets are *extensional entities* in that it is sufficient for two sets to be identical that they have the same members, and they have the same members in all possible worlds. Classes are *intensional entities*—they can *change members* across different possible worlds and just because two classes have the same members over a domain does not make them the same class.

[*]Fachbereich Philosophie, University of Konstanz. E-mail: neil.barton@uni-konstanz.de.

-If we have a non-potentialist outlook, the intensional account doesn't seem to buy us any difference between sets and classes.

-For a potentialist, however, we *can* think of classes as intensional entities.

-But how should we handle talk of sets and intensions for the potentialist?

Main Aims.

–(1.) Introduce some of the existing work here.

–(2.) Provide a semantics for potentialist sets and intensions.

–(3.) Argue that the kind of potentialism in play affects the logic you get out (in particular by how much it extends intuitionistic logic).

Here's the plan:

-§1 Potentialist Systems, Sets, and Intensions we ha -§2 Potentialist Semantics

-§3 Relationship to category theory

-§4 Conclusions and open questions

1 Potentialist Systems, Sets, and Intensions

-As we'll see, there are various ways in which potentialists might want to talk about sets.

-There are various kinds of potentialism we can consider.

-We might think of the sets as being brought into existence via the introduction of powersets, or by adding forcing generics, and so on...

Definition 1. [Hamkins and Linnebo, 2018] A *potentialist system* (for sets) is a pair $\mathbb{S} = (S, \leqslant_{\mathbb{S}})$, where S is a collection of structures of a certain kind and $\leqslant_{\mathbb{S}}$ is a refinement of the substructure relation.

-The potentialist systems that we will be concerned with in this paper are the following.

-For metamathematical ease we will fix some initial countable transitive model $M \models$ ZFC:

Definition 2. *The Rank Potentialist System* is the following potentialist system:

(i) $S = \{V_\beta^M | \beta \in On^M\}$

(ii) $V_\alpha^M \leqslant_{\mathbb{S}} V_\beta^M$ iff $\alpha \leqslant \beta$.

Definition 3. *The Generic Multiverse Potentialist System* is the following potentialist system:

(i) $S = \{M[G] | \text{"}G \text{ is } M\text{-generic"}\}$

(ii) $M[G] \leqslant_{\mathbb{S}} M[H]$ iff $M[H]$ is a forcing extension of $M[G]$.

Definition 4. *The Steel Multiverse Potentialist System* is the following potentialist system:

(i) $S = \{M[G \upharpoonright \alpha] | \text{"}G \text{ is } Col(\omega, < Ord^M)\text{-generic and } \alpha \in Ord^M\text{"}\}$

(ii) $M[G] \leqslant_{\mathbb{S}} M[H]$ iff $M[H]$ is a forcing extension of $M[G]$.

Definition 5. *The Countable Transitive Potentialist System* is the following potentialist system:

(i) $S = \{N| \text{ "}N \text{ is a countable transitive model of ZFC containing } M\text{"}\}$

(ii) $N \leqslant_{\mathbb{S}} N'$ iff $N \subseteq N'$

Definition 6. *The Actualist (Potentialist) System* is the following potentialist system:

(i) $S = \{M\}$

(ii) $M \leqslant_{\mathbb{S}} M$ iff $M = M$

-We now want to add in intensions.

-These can be thought of as functions from S to $\bigcup_{W \in S} W$ (note that since everything I'm considering is transitive, $\bigcup_{W \in S} W$ is well defined).

-For the sake of ease, I want to consider only those intensions which have ranges *contained* in the relevant worlds:

Definition 7. A function $f : S \rightarrow \bigcup_{W \in S} W$ is $\mathbb{S}$-*bounded* iff $f(W) \subseteq W$ for every $W \in S$.

Definition 8. An *intensional* potentialist system is a triple $\mathbb{S} = (S, \leqslant_{\mathbb{S}}, \mathcal{X})$, where $\mathcal{X}$ is a collection of $\mathbb{S}$-bounded functions.

-Two further definitions are going to be of use later:

Definition 9. A class f is *monotone* or *montonic* iff $x \in f(W)$ implies that $x \in f(W')$ for all $W' \geqslant_{\mathbb{S}} W$.

-Or a stronger condition:

Definition 10. A class f is *stable* iff:

(i) it is monotone, and

(ii) $x \notin f(W)$ and $x \in W$ implies that $x \notin f(W')$ for all $W' \geqslant_{\mathbb{S}} W$.

2 Potentialist Semantics

-We have the *modalised* language of set theory $\mathscr{L}_\in^\Diamond$, that is just $\mathscr{L}_\in$ except with the modal operators $\Diamond$ and $\Box$ added.

Definition 11. We define $(S, \leqslant_\mathbb{S})$ *satisfying* ϕ *at* W *under* v via the following recursive clauses:

(i) $W \Vdash_v x \in y$ iff $v(x) \in v(y)$

(ii) $W \Vdash_v \neg\phi$ iff $W \nVdash_v \phi$

(iii) $W \Vdash_v \phi \vee \psi$ iff $W \Vdash_v \phi$ or $W \Vdash_v \psi$

(iv) $W \Vdash_v \exists x\phi$ iff $W \Vdash_{v_1} \phi$ for some valuation v_1 that is like v except possibly on x, but where $v_1(x)$ is a member of W

(v) $W \Vdash_v \forall x\phi$ iff $W \Vdash_{v_1} \phi$ for every valuation v_1 that is like v except possibly on x, but where $v_1(x)$ is a member of W

(vi) $W \Vdash_v \Box\phi$ iff every $W' \geqslant_\mathbb{S} W$ is such that $W' \Vdash_v \phi$

(vii) $W \Vdash_v \Diamond\phi$ iff some $W' \geqslant_\mathbb{S} W$ is such that $W' \Vdash_v \phi$

-We have the usual *non-modal* language of set theory $\mathscr{L}_\in$, that has variables for sets and a single non-logical symbol $\in$.

Definition 12. A potentialist system $\mathbb{S}$ of $\mathscr{L}$-structures provides a potentialist account of a particular $\mathscr{L}$-structure N, if every world $W \in S$ is a substructure of N and furthermore, for every such W and every individual $a \in N$, there is a world $U \in S$ accessed by W with $a \in U$. ([Hamkins and Linnebo, 2018])

-We can now think of translating between our modal and non-modal theory as follows. To go from $\mathscr{L}_\in$ to $\mathscr{L}_\in^\Diamond$, replace every occurrence of $\exists$ with $\Diamond\exists$ and each occurrence of $\forall$ with $\Box\forall$ (denote the translation of ϕ by $\phi^\Diamond$).

Theorem 13. *[Linnebo, 2013], [Hamkins and Linnebo, 2018] (Model-Theoretic Mirroring) If $\mathbb{S}$ provides a potentialist account of a structure N, then truth in N is equivalent to potentialist truth at the worlds of W. Namely, for any $\mathscr{L}$-formula ψ and any a_0,..,$a_n \in N$, we have*

$$N \models \psi(a_0, ..., a_n) \leftrightarrow W \models \psi^\Diamond(a_0, .., a_n)$$

for any $W \in S$ in which the individuals $a_0, ..., a_n$ exist.

-We have the *non-modal* language of set theory with *intensions* $\mathscr{L}_{\in,\eta}$.

—We add variables C_0, ..., C_n, ... for intensions.

—We add a non-logical predicate η to hold between set-variables and intension variables.

-We then have the *modal* language of of set theory with *intensions* $\mathscr{L}^{\Diamond}_{\in,\eta}$.

-We then define satisfaction on a valuation as before:

Definition 14. We define $(S, \leqslant_{\mathbb{S}}, \mathcal{X})$ *satisfying* ϕ *at* W *under* v via the following recursive clauses:

(i) $W \Vdash_v x\eta Y$ iff $v(x) \in v(Y)(W)$

(ii) All the previous clauses for $\mathscr{L}^{\Diamond}_{\in}$ and expected compositional clauses.

(iii) $W \Vdash_v \exists X\phi$ iff $W \Vdash_{v_1} \phi$ for some valuation v_1 that is like v except possibly on X

(iv) $W \Vdash_v \forall X\phi$ iff $W \Vdash_{v_1} \phi$ for every valuation v_1 that is like v except possibly on X

Conjecture. I *strongly* expect that a mirroring theorem will be available for two-sorted class-theories. So where $(N, \mathcal{C})$ is a model of a theory in $\mathscr{L}_{\in,\eta}$ (e.g. NBG or NBG-Powerset) we can come up with potentialist accounts of truth in $(N, \in, \eta, \mathcal{C})$ via some $\mathbb{S} = (S, \leqslant_{\mathbb{S}}, \mathcal{X})$.

Idea. Restrict to stable intensions, and just chunk up any $C \in \mathcal{C}$ into $C \cap W$ at every $W \in \mathbb{S}$. We then say that $x\eta C$ (in $\mathscr{L}_{\in,\eta}$) iff there is a W such that $\Diamond x\eta C(W)$. Call this the **bivalent** interpretation.

Conjecture. Mirroring Theorems also come in syntactic flavours. I strongly suspect there is mirroring there too, but it's more difficult to show.

-This mirroring is good as far as it goes, but it's a bit annoying in certain respects.

-Whether or not a set is in a class becomes *too* coarse.

-Here's some example intensions (or at least conditions that will give you a $\mathbb{S}$-bounded function in the obvious way).

-e.g.1. $x = x$.

-e.g.2. x is constructible $Constructible(x)$

-e.g.3. x is countable $Countable(x)$

-$x = x$ is clearly stable.

-But $Constructible(x)$ (under rank potentialism) and $Countable(x)$ (under forcing potentialisms) are not.

-Instead of classical modal logic, let's adopt a free logic (something along these lines has already been done by Brauer-Linnebo-Shapiro).

-Let's consider the languages obtained when we just add in constants for every possible set and/or intension. I'll denote these by a bar over the script-L (e.g. $\bar{\mathscr{L}}^{\Diamond}_{\in}$).

-Assume the potentialist translation of $V = L$ under rank potentialism, and consider whether a set is constructible vs. whether it's self-identical.

-Consider whether a particular set x is countable under forcing potentialisms.

Slogan form of project. Ask not *if* a set x gets in to some class C, but **when**.

Definition 15. The *Boolean* intensional semantics is given by

$$[\![\phi]\!]_{\mathbb{S}} = \{W \in S | W \models \phi\}$$

-**Note:** I am fairly confident (though haven't checked the details) that this will validate classical logic.

-**Note:** You can always quotient down the Boolean interpretation into a bivalent one just by putting every non-empty $[\![\phi]\!]_{\mathbb{S}}$ onto 1 in the usual two-element Boolean algebra.

-This is all fine for the *liberal* potentialist modelling their attitude to truth, they think they have access to the whole modal process.

-But what about the *strict* potentialist?

-They think that things are *made* true.

-Instead of a Boolean approach to, it's much more natural to take a Kripke style semantics:

Definition 16. The *Kripke-style* semantics is given as follows:
$[\![\phi]\!]_{\mathbb{S}} = \{W \in S | \forall W' \geqslant_{\mathbb{S}} W, W' \models \phi\}$

-i.e. The truth-values correspond to "times in $\mathbb{S}$ from which ϕ is forevermore true".

-These will give us more fine-grained truth values that help us account for e.g. a particular set taking a long time to get into an intension.

-**Problem:** Get traction on these truth-values in determining properties of the logic.

3 Relationship to category theory

Definition 17. A *functor* $F : \mathscr{C} \to \mathscr{D}$ between categories $\mathscr{C}$ and $\mathscr{D}$ is a map that:

(i) Assigns every $\mathscr{C}$-object c a $\mathscr{D}$-object $F(c)$.

(ii) Assigns every $\mathscr{C}$-arrow $f : c_0 \to c_1$ a $\mathscr{D}$-arrow $F(f) : F(c_0) \to F(c_1)$ such that:

  (a) For every $\mathscr{C}$-object c, $F(Id_c) = Id_{F(c)}$, and

  (b) For every pair of $\mathscr{C}$-arrows $f : c_0 \to c_1$ and $g : c_1 \to c_2$, $F(g \circ f) = F(g) \circ F(f)$.

Definition 18. A *natural transformation* between functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{C} \to \mathscr{D}$ is a family of $\mathscr{D}$-arrows $\tau_c : F(c) \to G(c)$, one for each $\mathscr{C}$-object c, such that all squares of the following form commute (for all $\mathscr{C}$-objects a, b and all $\mathscr{C}$-arrows $f : a \to b$):

$$\begin{array}{ccc} F(a) & \xrightarrow{\tau_a} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow{\tau_b} & G(b) \end{array}$$

Definition 19. The *category of functors between $\mathscr{C}$ and $\mathscr{D}$* (often written $\mathscr{D}^{\mathscr{C}}$) is the category that has as objects all functors from $\mathscr{C}$ to $\mathscr{D}$ and as arrows all natural transformations between functors.

Fact 20. *If $\mathscr{C}$ is a small (i.e. set-sized) category, then* $\mathbf{Set}^{\mathscr{C}}$ *forms a topos.*

-In particular $\mathbb{S}$ is just such a category!

-Monotonic $\mathbb{S}$-bounded functions can be thought of as represented by (some) functors in $\mathbf{Set}^{\mathbb{S}}$.

-In particular, such a functor F will be a subobject of the following functor in $\mathbf{Set}^{\mathbb{S}}$:

Definition 21. $\mathit{dom}\text{-}S : S \to \mathbf{Set}$ is defined as the following functor:

(i) For every $W \in S$, $\mathit{dom}\text{-}S(W) = \bigcup S$

(ii) For every $f : W \to W'$ in $\mathbb{S}$, $\mathit{dom}\text{-}S(f) = Id_{\bigcup S}$

-Topoi come equipped with the following gadget:

Definition 22. If $\mathscr{C}$ is a category with a terminal object 1, then a *subobject classifier* for $\mathscr{C}$ is a $\mathscr{C}$-object Ω and a $\mathscr{C}$-arrow $\mathit{true} : 1 \to \Omega$ (sometimes written $\top : 1 \to \Omega$) with:

(Ω-Axiom) For every monic $\tau : F \rightarrowtail G$ there is exactly one $\mathscr{C}$-arrow χ_τ (the characteristic arrow of τ) such that the following diagram is a pullback square:

$$\begin{array}{ccc} F & \overset{\tau}{\rightarrowtail} & G \\ {\scriptstyle !^1_F}\downarrow & & \downarrow{\scriptstyle \chi_\tau} \\ 1 & \underset{\mathit{true}}{\longrightarrow} & \Omega \end{array}$$

The character $\chi_\tau : G \to \Omega$ of τ is defined at each W, and has components $(\chi_\tau)_W : G(W) \to \Omega(W)$, which in turn are defined coordinatewise:

for each $x \in G(W), (\chi_\tau)_W(x) = \{f : W \to W' | G(f)(x) \in F(W')\}$

$\chi_\tau(x)$ can be thought of as denoting the first time N (according to $\mathbb{S}$) at which x is determined to be in the subobject F.

Definition 23. For $x \in \bigcup S$ (i.e. the whole domain of our potentialist system) and monotonic intensional class f determining a functor $F : \mathbb{S} \to \mathbf{Set}$ we assign the sentence $x \in F$ (thought of a propositional variable) to the truth value $Val_{x\eta F} : 1 \to \Omega$ in $\mathbf{Set}^{\mathbb{S}}$ that picks, at a world W the following element of $\Omega(W)$:

$$\{f : W \to W' | x \in f(W')\}$$

i.e. $Val_{f \in x}(W) : 1(W) \to \Omega(W)$ is defined component-wise by mapping the unique element of $1(W)$ to the set of those (arrows to) worlds W' for which $x \in f(W')$.

Definition 24. Let $\perp$ be the character of $!^0_1 : 0 \to 1$. Then $\neg : \Omega \to \Omega$ is the character of $\perp$ in $\mathscr{E}$, i.e. $\neg = \chi_\perp$.

Definition 25. $\wedge : \Omega \times \Omega \to \Omega$ is the character in $\mathscr{E}$ of the product arrow $\langle \top, \top \rangle : 1 \to \Omega$.

Definition 26. $\vee : \Omega \times \Omega \to \Omega$ is the character of the image of the arrow $[\langle \top_\Omega, Id_\Omega \rangle, \langle Id_\Omega, \top_\Omega \rangle] : \Omega + \Omega \to \Omega \times \Omega$

Definition 27. Let $e : \leqslant \rightarrowtail \Omega \times \Omega$ be the equaliser of $\wedge : \Omega \times \Omega \to \Omega$ and $pr_1 : \Omega \times \Omega \to \Omega$ (where pr_1 is the projection onto the first coordinate of $\Omega \times \Omega$). Then $\to : \Omega \times \Omega \to \Omega$ is the character of e (i.e. $\to = \chi_e$).

Definition 28. Given a topos $\mathscr{E}$, let $\mathscr{E}(1, \Omega)$ denote the collection of all arrows from the terminal object to Ω. A $\mathscr{E}$-valuation is a function Val that assigns each propositional variable P a truth value $f : 1 \to \Omega$. Val in turn determines a value for each propositional well-formed formula $\phi, \psi, ...$ etc. recursively via the following compositional clauses:

(i) $Val(\neg\phi) = \neg \circ Val(\phi)$

(ii) $Val(\phi \wedge \psi) = \wedge \circ \langle Val(\phi), Val(\psi) \rangle$

(iii) $Val(\phi \vee \psi) = \vee \circ \langle Val(\phi), Val(\psi) \rangle$

(iv) $Val(\phi \to \psi) = \to \circ \langle Val(\phi), Val(\psi) \rangle$

Furthermore, we say that ϕ is $\mathscr{E}$-valid (or $\mathscr{E} \models \phi$) iff every $\mathscr{E}$ valuation Val, $Val(\phi) = \top : 1 \to \Omega$.

-This coincides with the propositional variables and connectives with the account given earlier.

-We already knew that Kripke-semantics over some $\mathbb{P}$ and valuations from $\mathbf{Set}^{\mathbb{P}}$ coincide.

-This gives us traction at least on the propositional case.

Theorem 29. *[Dummett, 1959], [Segerberg, 1968] (The Dummett-Segerberg Theorem) Suppose that $\mathbb{S}$ is an infinite linearly ordered poset. Then $\mathbf{Set}^{\mathbb{S}} \models \phi$ iff ϕ is a theorem of intuitionistic propositional logic with all classical tautologies of the form $(\psi_0 \rightarrow \psi_1) \vee (\psi_1 \rightarrow \psi_0)$ added (so called Dummett's Logic).*

Fact 30. *For partial order $\mathbb{P}$, the topos $\mathbf{Set}^{\mathbb{P}}$ satisfies the weak excluded middle law $\neg\phi \vee \neg\neg\phi$ if $\mathbb{P}$ is confluent, i. e. for all $x \leqslant_{\mathbb{P}} y, y'$, there is a z with $y, y' \leqslant_{\mathbb{P}} z$.*

-It's interesting that the link is **so close** between the objects of $\mathbf{Set}^{\mathbb{S}}$ and intensional classes.

-**Note:** It's not completely clear to me how to get traction on this in the first-order case.

4 Conclusions and open questions

Question/Conjecture. Properly explore the mirroring phenomenon in this context.

Question. How to generalise to first-order (Ω-sets)?

Question. What about different kinds of class beyond stable/monotonic?

Definition 31. An intensional class f is *capricious* iff for every set $x \in \bigcup S$ and every world W with $x \in W$, there are $W_1, W_2 \geqslant_{\mathbb{S}} W$ such that $x \notin f(W_1)$ and $x \in f(W_2)$.

(i.e. f cannot make up it's mind)

Definition 32. A class f is *eventually convergent* iff there are worlds $W_1 \leqslant_{\mathbb{S}} W_2 \leqslant_{\mathbb{S}} W_3$ with $x \in f(W_2)$ taking the opposite (classical) value from $x \in f(W_1)$ and $x \in f(W_3)$, but where there is a world $W_4 \geqslant_{\mathbb{S}} W_3$ such that for all $W_5 \geqslant_{\mathbb{S}} W_4$, $x \in f(W_5)$ or $x \notin f(W_5)$ uniformly.

(i.e. one of $x \in f$ or $x \notin f$ is true at every world past some point)

-e.g. *uncountable set* (under forcing potentialism), *non-constructible set* under rank potentialism (with $V = L$).

Question. How to handle non-transitive contexts?

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