Mathematical Gettier Cases

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- OK, so what is a Gettier case?
- This is a case where an agent has a justified and true belief, but (intuitively) not knowledge.
- e.g. Fake barn country: I'm travelling through the countryside and see what I think are many barns around me. Pointing at one, I say "That's a barn". As it turns out, it is a barn. However, unbeknowst to me, they are filming the $(n+1)^{th}$ film of *Fast and the Furious* (for suitably large n) and there are many facsimile papier-mâché barns (Vin Diesel is going to jump cars over them later). So whilst what I said was true and justified, I got epistemically lucky.

Main Question.

Can mathematical justification (i.e. the kind of justification for claims that proofs provide in mathematics journals) be Gettiered and what implications might this have?

Introduction

- There's at least one kind of puzzle for mathematical Gettier cases.
- The standard of mathematical justification is proof from the axioms (right?)...
- But the axioms are true and the rules of proof preserve truth...
- So how could one ever have a true but epistemically lucky justified belief?
- Proof from the axioms already entails that we're not lucky!
- Even more acute when we consider the usual templates for Gettier cases (e.g. Zagzebski).

Strategy

Target.

The possibility (and in some cases actuality) of mathematical Gettier cases indicates some important upshots for the practice of mathematics.

- §1 Simil-Proofs
- §2 Axiom Selection
- §3 Black-box Lemmas
- §4 Internalist and Externalist Criteria
- §5 Upshots for Mathematical Practie

§1 Simil-Proofs

- Unfortunately for us proofs just aren't always axiomatic proofs.
- Mathematics journals really contain (as well as discussion of significance of theorems etc.):

Definition (Philosophical).

An argument is a Simil-Proof (*SP*) when it is [(i)] shareable, and [(ii)] some agents who have judged all its parts to be correct as a result of checking accept it as a proof. Moreover, [(iii)] the argument broadly satisfies the standards of acceptability of the mathematical community to which it is addressed. ([De Toffoli, 2020], p. 13)

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- Of course Simil-Proofs are easily Gettiered (make a non-trivial error that doesn't get caught).
- But there the agent clearly is (somewhat) epistemically culpable.
- Are there cases where this doesn't follow?
- There's some easy examples (e.g. hardware failure).
- But they don't really tell us anything non-obvious.
- There are (at least) two kinds that do.

§2 Axiom Selection

- It's been known for quite some time that selecting axioms for certain subject matters can be tricky.
- Set theory is a good example here, since we have many natural ways we might extend our usual set theory (**ZFC**).
- To take two examples, there are justifications for both the Proper Forcing Axiom and $V =$Ultimate-L (it doesn't matter what these say for the philosophical point).
- These two theories agree on some points (e.g. Projective Determinacy) but disagree on others (e.g. the Continuum Hypothesis).

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- Suppose then that I'm a member of the PFA-Lykovs who disappeared into the wilderness sometime around 1930.
- I believe PFA in virtue of these good justifications.
- I then prove some ϕ on this basis.
- But as it turns out, $V =$Ultimate-L is true.
- Suppose further that $V =$Ultimate-L agrees on ϕ.
- It looks like I have justified true belief in ϕ, but the justification was lucky.
- My false but well-justified belief has lead me to truth.
- It doesn't seem like I've done anything epistemically blameworthy here (I'm just ignorant of alternatives).

§3 Black-box Lemmas

- In the last example, I needed a false lemma to get things going.
- An issue here concerns how much one should have understood of the proof one provides.
- Mathematics is replete with the use of lemmas that are often used as 'black boxes'.
- This is an important practice for allowing mathematics to move forward as a community.

1. It's unreasonable to expect mathematicians to follow through every lemma they rely on.
2. Using lemmas from subject matters with which one is not familiar is very fruitful.

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- This facilitates a Gettier case.
- Suppose I use a published, well-peer-reviewed lemma from a different field as a black box in a proof.
- As it turns out, the proof of that lemma is flawed (but the lemma is nonetheless true).
- It seems then that I do have justified true belief—I can perfectly well understand all the elements of my proof (even if I don't understand the proof of the lemma).
- But I don't know, it could easily have been the case that I had ended up refraining from believing my theorem or believing its negation.
- There are actual examples of the flavour I describe (e.g. Dehn's Lemma was thought proved in 1910, a flaw was found in 1929, and it was finally proved only in 1957, the Four Colour Theorem was thought proved 1879–1891, examples can be multiplied).

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§4 Externalist and Internalist Criteria

There are at least two criteria that are relevant to these examples:

The Externalist Criterion.

Do the mathematical facts explain the steps of my simil-proof, or is something else at play?

The Internalist Criterion.

How well do I understand the steps and dependencies of my simil-proof?

In these cases, though the agents are not really epistemically blameworthy something has gone wrong with respect to one or more of these criteria.

§5 Upshots for Mathematical Practice

- I think these cases have some concrete payoffs for how we go about doing mathematics.
- We need to do all we can in mathematics to ensure that the relevant explanatory links are in place.
- We need to understand the dependencies of simil-proofs as well as possible.
- Some are clear (e.g. do as much background work as possible, be conscientious in work, proof-assistants look great!)
- 1. Folklore theorems are often bad.
- An example of Rittberg, Tanswell, and Van Bendegem: A young topos theorist (Olivia Caramello) had extreme trouble trying to publish her 'duality theorem' in topos theory, for the reason it was 'folkloric'.

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"Although considered "folkloric" by some experts, the result does not appear in the literature. I had believed that one could directly deduce it from the theory of classifying toposes of Makkai and Reyes. [There was] an aspect of Caramellos proof which I had missed... Surprised by this observation, I tried to exhibit the "folkloric" proof that I thought I had of this theorem. With my great astonishment, it took me a night of work to construct a proof based on my knowledge of the subject, and the proof depended only partially on Makkai-Reyes theory!(André Joyal, in a public letter to Olivia Caramello)

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- Note that a substantial part of Caramello's result, even if you think that the result is relatively easy, was in clarifying the underlying logical (and hence presumably explanatorially informative) links and dependencies.
- It seems then with folklore theorems the following dichotomy obtains:

1. Either the proof really is just a tedious and/or unilluminating exercise, and so it can be flagged as such (possibly with a hint on how it should go).
2. Or, it's at least somewhat non-trivial, in which case it's worth having it in the literature for inspection and drawing out links. (e.g. as graduate theses—something that has improved recently in set theory).

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- 2. Re-proving theorems with different proofs is really important.
- This is a common and accepted practice in mathematics.
- The mathematical usefulness of this practice is clear.
- However, the examples showcase that there's substantial epistemic payoff too: Unfolding the links in different areas makes it more likely that our beliefs are explained by the relevant mathematical facts.

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- 3. We should be inclined towards a methodological pluralism concerning mathematics (including foundations!).
- Using multiple different perspectives facilitates the drawing out of explanatory links in different contexts (see, also, [Barton, 2017]).
- In particular, the phenomenon of convergence is an important for epistemic reasons.
- Insisting on one theory (possibly foundational) at the expense of others blinkers us here.

Conclusions

- Whilst Gettier-cases are often relatively abstract, the considerations of how they can arise is informative for the practice of mathematics.
- Really though, this should motivate a much wider and interdisciplinary study.

Question.

Who are the epistemic agents here? Is it me, and you, and all the other mathematical beings individually? Or is the interesting epistemic agent the community?

Question.

Are there ways of empirically analysing to what degree mathematical communities are conforming to particular normative standards?

Thanks! Discussion!

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