Intensional Classes and Intuitionistic Topoi

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Introduction

- The problem of proper classes has been with us for a long time.
- There are various conditions e.g. $x \notin x$, that fail to define sets (on pain of contradiction!).
- The iterative conception of set (sets are formed in stages by forming all available sets at successor stages) prohibits the existence of these collections as sets.

Introduction

However:

(1.) I seem to be able to say true things about these classes, e.g. $\{x|x \notin x\} = \{x|x = x\}$

(2.) They seem to play an important role in set-theoretic practice:

- e.g. 1. κ is *measurable* iff κ is the critical point of a non-trivial elementary embedding $j : V \rightarrow M$,
- e.g. 2. Kunen's Theorem: There is no non-trivial elementary embedding $j : V \rightarrow V$.

So how should we interpret talk of these entities? In particular:

Difference Question.

How are proper classes different from sets?

Introduction

One response:

The Intensional Account of Classes

Sets are extensional entities in that it is sufficient for two sets to be identical that they have the same members. Classes are intensional entities—they can change members.

But how can classes change members if mathematical objects exist out of necessity?

Potentialism.

'The' universe of set theory is never fully completed, but rather unfolds gradually as parts of it increasingly come into existence or become accessible to us. ([Hamkins and Linnebo, 2018], p. 1)

Introduction

Main Question.

How can we formalise this talk of intensional classes under set-theoretic potentialism? What does this tell us about reasoning with them?

Main Claims.

1. We can formalise (some) intensional classes using category theory.
2. This suggests (at this stage) that their logic should be non-classical, and the kind non-classicality is linked variety of potentialism in play.

Secondary aim for the talk.

Convince you that category theory has some interesting features when it comes to modal variation.

Outline

Main Claims.

1. We can formalise (some) intensional classes using category theory.
2. This suggests (at this stage) that their logic should be non-classical, and the kind non-classicality is linked variety of potentialism in play.

Here's the plan:

- §1 Potentialist systems and intensional classes
- §2 A whirlwind introduction to topos theory and functor categories
- §3 Application to potentialism
- §4 Conclusions and open questions

§1 Potentialist systems and intensional classes

- Given the idea the universe of set theory 'grows', a natural way is to think of this modally.
- Given any sets, there could have been more.
- There are different ways we could view the sets growing:
  1. We could add ranks (Rank Potentialism).
  2. We could add subsets via forcing (Forcing Potentialism).
  3. We could look at a situation in which we can (among other things) add ranks and add subsets (Countable Transitive Model Potentialism).

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- Of course, our standard set theories (e.g. **ZFC**) don't have modal operators or variables for worlds etc., so a standard move is to engage in a kind of modelling procedure.
- We fix a background universe of set theory V.
- And then fix various potentialist systems within V, conceived of as sets with relations between them.
- Studying these systems (one might think) tells us about what goes on in (potentialist) 'reality' (assuming one is a potentialist).
- This is similar to how if one believes that there is a single maximal non-potential universe of sets V, one can still learn about that universe by studying set-sized models in V.[1]

[1] For an in depth look at this idea. See [Barton, 2020b] and [Antos et al., F].

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- So fix an ambient universe of set theory V and some countable transitive model $M \models \mathrm{ZFC}$

Definition.

The Rank Potentialist System:

(I) $S = \{V_\beta^M | \beta \in On\}$

(II) $V_\alpha^M \leq_S V_\beta^M$ iff $\alpha \leq \beta$.

(III) $\Diamond\phi =$ "True in the current world or some larger V_β^M"

Definition.

The Forcing Potentialist System :

(I) $S = \{M[G] | \text{“}G \text{ is } M\text{-generic”}\}$

(II) $M[G] \leq_S M[H]$ iff $M[H]$ is a forcing extension of $M[G]$.

(III) $\Diamond\phi =$ “True in some forcing extension”

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Definition.

The Countable Transitive Potentialist System is the following potentialist system:

(i) $S = \{N| \text{ ``}N \text{ is a countable transitive model of ZFC''}\}$

(ii) $N_1 \leq_S N_2$ iff $N_1 \subseteq N_2$.

(iii) $\Diamond\phi =$ "True in the current world or some larger countable transitive model of ZFC"

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- We want to consider some formalisation that allows us to move through the worlds and pick up more sets depending on where we go.

§2 A whirlwind introduction to topos theory and functor categories

- Any introduction here is going to be hopelessly brief and/or too fast.
- But, we can get a feel for the key ideas, and see why topos theory might be interesting for discussing modality.
- The important points:

1. Categories give you a way of handling mapping one object to another.
2. A topos is a particular kind of category that has enough structure to provide truth values and a way of linking this to a generalisation of the idea of subset.
3. A functor category is a way of looking at transformations between mappings from one category to another.

In the end we'll see that intensional classes are naturally represented by certain functors within a functor category, and the properties of the ambient topos are what gives us the non-classicality.

DEFINITION.

A *category* $\mathscr{C}$ comprises:

(I) A collection of objects ($\mathscr{C}$-objects).

(II) A collection of arrows ($\mathscr{C}$-arrows) between the objects

(III) Operations $dom(f)$ and $cod(f)$ assigning to any arrow f its domain or codomain.

(IV) An operation $\circ$ assigning to each pair of arrows $f : a \to b$, $g : b \to c$ the *composition* $f \circ g : a \to c$ that is *associative* i.e. $f \circ (g \circ h) = (f \circ g) \circ h$

(V) An assignment to each $\mathscr{C}$-object b a $\mathscr{C}$-arrow $Id_b : b \to b$ (the *identity arrow* on b) such that for any $\mathscr{C}$-arrows $f : a \to b$ and $g : b \to c$, $Id_b \circ f = f$ and $g \circ Id_b = g$.

Two notions are going to be really important for developing the notion of a topos and associated idea of semantics.

Definition.

An object 1 is *terminal* in $\mathscr{C}$ iff for any $\mathscr{C}$-object a there is exactly one $\mathscr{C}$-arrow $!^1_a : a \to 1$. (In **Set**, these are singletons $\{a\}$.)

Definition.

A $\mathscr{C}$-object 0 is *initial* in $\mathscr{C}$ iff for any $\mathscr{C}$-object a there is exactly one arrow $!^0_a : 0 \to a$. (In **Set**, $\emptyset$ is the unique initial object.)

- We're going to use truth values classify subset-like arrows.
- Subsets can be viewed as maps in set, specifically certain injections.

Definition.

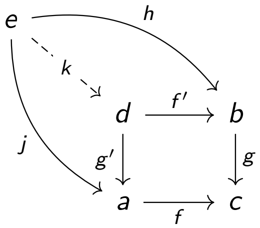
A $\mathscr{C}$-arrow $f : a \to b$ is *monic* (often written $f : a \rightarrowtail b$) iff for any two arrows $g : c \to a$ and $h : c \to a$, $f \circ h = f \circ g$ implies $h = g$.

- In **Set**, the monics are exactly the injections.
- Before we give our 'subobject classifier' we need one last definition.

A *pullback* of a pair of $\mathscr{C}$-arrows $f : a \to c$ and $g : b \to c$ is a pair of $\mathscr{C}$-arrows $g' : d \to a$ and $f' : d \to b$ such that:

(i) $f \circ g' = g \circ f'$

(ii) Whenever we have two $\mathscr{C}$-arrows $h : e \to a$ and $j : e \to b$ with $f \circ h = g \circ j$, there is exactly one $\mathscr{C}$-arrow $k : e \dashrightarrow d$ such that the following diagram commutes:



In **Set**, $d = \{\langle x, y\rangle | x \in a \wedge y \in b \wedge f(x) = g(y)\}$, and $f'(\langle x, y\rangle) = y$ and $g'(\langle x, y\rangle) = x$.

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If $\mathscr{C}$ is a category with a terminal object 1, then a *subobject classifier* for $\mathscr{C}$ is a $\mathscr{C}$-object Ω and a $\mathscr{C}$-arrow $true : 1 \to \Omega$ (sometimes written $\top : 1 \to \Omega$) such that:

(Ω-Axiom) For every monic $f : a \rightarrowtail b$ there is exactly one $\mathscr{C}$-arrow χ_f (the characteristic arrow of f) such that the following diagram is a pullback square:

$$\begin{array}{ccc} a & \overset{f}{\rightarrowtail} & d \\ {\scriptstyle !^1_a}\downarrow & & \downarrow{\scriptstyle \chi_f} \\ 1 & \underset{true}{\longrightarrow} & \Omega \end{array}$$

In **Set**, *true* picks out the top truth value (i.e. 1) and χ_f classifies whether a set is in or out of the subset of d picked out by f.

We can now put the pieces together to give the definition of a topos, a category with enough structure to classify subobjects and give truth values (we'll see a little how to the truth stuff later):

Definition.

A *topos* (or *elementary topos*) is a category $\mathscr{E}$ such that:

(I) $\mathscr{E}$ has a terminal object and pullbacks.

(II) $\mathscr{E}$ has an initial object and ~~pushouts~~.

(III) ~~$\mathscr{E}$ has exponentiation~~.

(IV) $\mathscr{E}$ has a subobject classifier.

We're going to represent intensional classes by the following kind of object:

Definition.

A *functor* $F : \mathscr{C} \to \mathscr{D}$ between categories $\mathscr{C}$ and $\mathscr{D}$ is a mapping that:

(I) Assigns every $\mathscr{C}$-object c a $\mathscr{D}$-object $F(c)$.

(II) Assigns every $\mathscr{C}$-arrow $f : c_0 \to c_1$ a $\mathscr{D}$-arrow $F(f) : F(c_0) \to F(c_1)$ such that:

- (A) For every $\mathscr{C}$-object c, $F(Id_c) = Id_{F(c)}$, and
- (B) For every pair of $\mathscr{C}$-arrows $f : c_0 \to c_1$ and $g : c_1 \to c_2$, $F(g \circ f) = F(g) \circ F(f)$.

We need a notion of mapping between the functors (so we can start classifying subobjects).

Definition.

A *natural transformation* between functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{C} \to \mathscr{D}$ is a family of $\mathscr{D}$-arrows $\tau_c : F(c) \to G(c)$, one for each $\mathscr{C}$-object c, such that all squares of the following form commute (for all $\mathscr{C}$-objects a, b and all $\mathscr{C}$-arrows $f : a \to b$):

$$\begin{array}{ccc} F(a) & \xrightarrow{\tau_a} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow{\tau_b} & G(b) \end{array}$$

Let $\mathscr{C}^{\mathscr{D}}$ be the category of functors from $\mathscr{D}$ to $\mathscr{C}$ that has functors from $\mathscr{D}$ to $\mathscr{C}$ as objects and natural transformations between functors as arrows. Then:

Fact.

If $\mathscr{C}$ is small (i.e. set-sized) then $\mathbf{Set}^{\mathscr{C}}$ forms a topos.

§3 Application to potentialism

The key observation that gets the interpretation going is the following two:

Observations.

(1.) All of the potentialist systems we talked about earlier can be thought of as partial orders.

(2.) Partial orders can themselves be thought of as categories.

So, for any of the potentialist systems we've considered, there's a functor category $\mathbf{Set}^{\mathbb{P}}$ that forms a topos in V.

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- Let's think about what this functor category does.
- A functor $F : \mathbb{P} \to \mathbf{Set}$ takes a world in $\mathbb{P}$ (i.e. the potentialist system)...
- ...and gives me back an extension (arrows in $\mathbb{P}$ can be mapped on to inclusions in **Set**).
- We have something intensional here!
- [Lawvere, 1975] described functor categories $\mathbf{Set}^{\mathbb{P}}$ as 'sets through time'.

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- We need to restrict the intensional classes and functors we consider to get what we want:
- We'll only consider classes that are monotonic. (i.e. only pick up members and do not lose members—I'm unsure how to handle non-monotonic classes with functors, it may well be known!)
- We need to ensure that the functors only pick up sets from the right worlds (say a functor $F : \mathbb{P} \rightarrow \mathbf{Set}$ is $\mathbb{P}$-*bounded* iff for every $w \in \mathbb{P}$, $F(w) \subseteq w$).
- We'll be interested in the $\mathbb{P}$-bounded functors in $\mathbf{Set}^{\mathbb{P}}$, that provide a representation of the monotonic intensional classes.

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What is the structure of the subobject classifier here?

- The terminal object is the constant functor $1 : \mathbb{P} \rightarrow \mathbf{Set}$ that maps every element of p onto $0 \in \{0\}$, and every $f : p \rightarrow q$ in $\mathbb{P}$ onto $Id_{\{0\}}$.
- A monic natural transformation $\tau : F \rightarrowtail G$ is composed of injections $\tau_p : F(p) \rightarrowtail G(p)$ in **Set**.
- We need to say what Ω and $\top : 1 \rightarrow \Omega$ are.

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The notion of a truth value is captured by:

Definition.

Given a $\mathbb{P}$-object p, we let S_p be the collection of all $\mathbb{P}$-arrows with domain p. A p-sieve is a subset $S \subseteq S_p$ such that S is closed under left composition, i.e. for every $f, g \in S_p$, if $f \in S$ then $g \circ f \in S$.

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- The arrow $\top : 1 \to \Omega$ is defined component-wise, and maps $0 \in 1(p) = \{0\}$ on to the largest p-sieve in $\Omega(p)$.
- The character $\chi_\tau : G \to \Omega$ of τ is defined at each p, and has components $(\chi_\tau)_p : G(p) \to \Omega(p)$, which in turn are defined coordinate-wise:

$$\text{for each } x \in G(p), (\chi_\tau)_p(x) = \{f : p \to q | G(f)(x) \in F(q)\}$$

Slogan:

Ask not if some set x in F is in G, but when.

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Proposal.

- Let $MULT : \mathbb{P} \rightarrow Set$ be the functor that maps every $p \in \mathbb{P}$ onto the union of the domains of our potentialist system (i.e. the set (in V) of all possible sets (in the sense of $\mathbb{P}$)).
- Consider a monotonic intensional class represented by a $\mathbb{P}$-bounded functor $F : \mathbb{P} \rightarrow Set$.
- A monic natural transformation $\tau : F \rightarrowtail MULT$ then represents how F's extension develops as we move through the potentialist system.
- Let η be the 'membership/application relation' that holds between sets and intensional classes.
- We should assign $x \eta F$ the p-sieve formalising when x becomes a member of F (in the sense of the subobject classifier).

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- On this basis, there's good reason to think that we should get non-classical truth-value assignments.
- e.g. Rank Potentialism: Consider the intension *if there is an inaccessible cardinal, then pick out the reals otherwise pick out* $\emptyset$. (We can come up with more interesting analogues of this related to definability.)
- Taking the modality seriously means that we do have to wait.

- e.g. Both Forcing Potentialism and Countable Transitive Model Potentialism:

Fact.

(Mostowski) Over a countable transitive model $M \models \text{ZFC}$, there are Cohen reals $M[G]$ and $M[H]$ such that there is no countable transitive model $N \models \text{ZFC}$ with height $Ord(M)$ and such that both $G, H \in N$

Consider the intension of *being a Cohen real*. There is branching (irrevocably for the Forcing Potentialist) and non-inevitability.

§4 Conclusions and open questions

So, what do I think I've established?

Main Conclusion.

We can represent monotonic intensional classes by functors. The truth values we ascribe to claims about membership should not just be 0 and 1 but rather intermediate values, and the kinds of values we get depend on the variety of potentialism being offered.

But there's a bunch of open questions here, and the analysis is very preliminary.

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First some mathematical ones:

1. What exactly is the logic validated by the relevant topos-theoretic semantics? (See Appendix!)
2. What does this look like when we generalise to first-order logic?
3. Is there a way of handling non-monotonic classes?

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But there are some really important philosophical questions to be addressed here too:

1. That last question is pertinent: Often we think of intensional classes as given by application conditions. If you have non-trivial monotonic classes, then you also have non-trivial non-monotonic classes (just look at the negation of the condition and hence the complement).
2. How well does this modelling fit intensional classes?
3. Is there a way to come up with an account of classes that restores classicality?
4. Should we even want classicality as potentialists?
5. More generally, how do these ideas generalise?
   - 5.1 Intensional classes in scientific discourse? (e.g. measurements)
   - 5.2 The general idea of using categories to represent mappings and deformations?[2]

[2] See [Landry, 2017] for a nice collection here and [Barton, 2020a] for a summary.

Thanks

Thanks for listening, I look forward to the comments!

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Antos, C., Barton, N., and Friedman, S.-D. (F).
Universism and extensions of V.
Forthcoming in *The Review of Symbolic Logic*.

Barton, N. (2020a).
Elaine Landry,* ed. Categories for the Working Philosopher.
Philosophia Mathematica, 28(1):95–108.

Barton, N. (2020b).
Forcing and the universe of sets: Must we lose insight?
Journal of Philosophical Logic, 49(4):575–612.

Dummett, M. (1959).
A propositional calculus with denumerable matrix.
The Journal of Symbolic Logic, 24(2):97–106.

Hamkins, J. D. and Linnebo, O. (2018).
The modal logic of set-theoretic potentialism and the potentialist maximality principles.
to appear in Review of Symbolic Logic.

Landry, E., editor (2017).
Categories for the Working Philosopher.
Oxford University Press.

Lawvere, F. W. (1975).
Continuously variable sets; algebraic geometry = geometric logic.
In Rose, H. and Shepherdson, J., editors, Logic Colloquium '73, volume 80 of Studies in Logic and the Foundations of Mathematics, pages 135–156. Elsevier.

Segerberg, K. (1968).
Propositional logics related to Heyting's and Johansson's.
Theoria, 34(1):26–61.

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Appendix: Logic in $\mathbf{Set}^{\mathbb{P}}$

Definition

- Let $\bot$ be the character of $!_1^0 : 0 \to 1$. Then $\neg : \Omega \to \Omega$ is the character of $\bot$ in $\mathscr{E}$, i.e. $\neg = \chi_\bot$.
- $\wedge : \Omega \times \Omega \to \Omega$ is the character in $\mathscr{E}$ of the product arrow $\langle \top, \top \rangle : 1 \to \Omega$.
- $\vee : \Omega \times \Omega \to \Omega$ is the character of the image of the arrow $[\langle \top_\Omega, Id_\Omega \rangle, \langle Id_\Omega, \top_\Omega \rangle] : \Omega + \Omega \to \Omega \times \Omega$
- Let $e : \leq \rightarrowtail \Omega \times \Omega$ be the equaliser of $\wedge : \Omega \times \Omega \to \Omega$ and $pr_1 : \Omega \times \Omega \to \Omega$ (where pr_1 is the projection onto the first coordinate of $\Omega \times \Omega$). Then $\to : \Omega \times \Omega \to \Omega$ is the character of e (i.e. $\to = \chi_e$).

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Definition

Let $\mathscr{E}(1, \Omega)$ denote the collection of all arrows from the terminal object to Ω. A $\mathscr{E}$-valuation is a function *Val* that assigns each propositional variable *P* a truth value $f : 1 \to \Omega$. *Val* in turn determines a value for each propositional well-formed formula $\phi, \psi, \ldots$ etc. inductively via the following compositional clauses (we abuse notation slightly and allow *Val* to denote the whole assignment too):

(I) $Val(\neg\phi) = \neg \circ Val(\phi)$

(II) $Val(\phi \wedge \psi) = \wedge \circ \langle Val(\phi), Val(\psi)\rangle$

(III) $Val(\phi \vee \psi) = \vee \circ \langle Val(\phi), Val(\psi)\rangle$

(IV) $Val(\phi \to \psi) = \to \circ \langle Val(\phi), Val(\psi)\rangle$

Furthermore, we say that ϕ is $\mathscr{E}$-valid (or $\mathscr{E} \models \phi$) iff every $\mathscr{E}$ valuation *Val*, $Val(\phi) = \top : 1 \to \Omega$.

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Theorem

[Dummett, 1959], [Segerberg, 1968] Suppose that $\mathbb{P}$ is an infinite linearly ordered poset. Then $\mathbf{Set}^{\mathbb{P}} \models \phi$ iff ϕ is a theorem of intuitionistic propositional logic with all classical tautologies of the form $(\psi_0 \to \psi_1) \vee (\psi_1 \to \psi_0)$ added (so called Dummett's Logic).

This means that the ambient topos for the Rank Potentialist System is non-classical but has several classical tautologies. These are lost in the Countable Transitive Model and Forcing Potentialist systems. Two questions:

1. Do these properties of the ambient topos transfer to the $\mathbb{P}$-bounded functors?
2. Does the directedness of the Countable Transitive Model Potentialist's system change anything (in contrast to the Forcing Potentialist)?