Algebraic Levels in Mathematical Structuralism

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Introduction

- This talk concerns how model theory can inform our thinking concerning the notion of structure in philosophy.
- But first, I want to start with some thanks.
  1. Thank you for the opportunity to talk!
  2. Thanks to Moritz Müller for teaching a fab model theory course back while I was at the KGRC.
  3. Thanks to John Baldwin and Andrés Villaveces for many patient discussions about model theory.
  4. Thanks to Tim Button for letting me include some (early stage) joint work I'll gesture to later.

Introduction

Let's start with the following:

Assumption.

(The Structuralist Assumption) The subject matter of mathematics is constituted by structures.

Introduction

Quite often, contemporary model theory is regarded as orthogonal to the study of mathematical structuralism:

> *The point is that the model-theoretic notion of structure takes as its starting point a domain of objects and is a construction (definition) within set theory with urelemente, or within pure set theory. insofar as the notion of mathematical object is philosophically problematic, appeal to this account begs the question. ([Isaacson, 2011], p. 26)*

But we should distinguish between model theory as providing an ontological foundation for structuralism and being informative for understanding how we talk about structure.

Introduction

Main Claim.

Model theory is important for understanding structuralism better. In particular, it helps us see that there are kinds of structure that are indeterminate (in certain respects).

- §1 Cardinality and Truth in mathematical structuralism
- §2 Philosophically modular theories and structures
- §3 Strongly minimal theories and structures
- §4 Comparing strongly minimal and determinate structures
- §5 Open problems

§1 Cardinality and Truth in mathematical structuralism.

- Let's start by making a distinction between mathematical concepts, discourses, formal theories, systems, and structures.
- e.g. Clock Arithmetic.

§1 Cardinality and Truth in mathematical structuralism

Two kinds of theory we encounter:

- Algebraic theories are those that do not have a single intended model up to isomorphism (e.g. the axioms for a group).
- Non-Algebraic theories are those that do have a single intended model up to isomorphism (e.g. the axioms for $(10, <)$ or PA_2).

There are two kinds of corresponding structure here:

- Determinate structures are ones all of whose exemplars are isomorphic (perhaps up to definitional equivalence) e.g. the number 10 under the less-than relation.
- Indeterminate structures are those that have non-isomorphic exemplars (perhaps up to definitional equivalence) e.g. the group structure.

§1 Cardinality and Truth in mathematical structuralism

- Many structuralists (e.g. [Hellman, 1989], [Shapiro, 1997], [Isaacson, 2011], [Leitgeb, 2020]) hold that the fundamental kind of sameness of structure a system can exhibit is through isomorphism (possibly modulo definitional equivalence).
- This bijection-based criterion has the following two consequences:

Truth.

What is true in base-level structures is fixed.

Cardinality.

The size of each fundamental structure is fixed.

Importantly indeterminate structures should just be understood as type-raising higher order properties, that can then be understood either nominalistically or as entities of a different kind.

§2 Philosophically modular structures

- Consider the following theory (that I could write in first-order logic): "I consist solely of independent two-cycles."
- This isn't a non-algebraic theory talking about a determinate structure.
- But it does have a particular structure as its base, with an instruction for how a model should be built up (just repeat!).

§2 Philosophically modular structures

Definition.

(Informal) A theory is philosophically modular iff it encodes:

- A template structure, and
- A precise set of instructions to build up a unique structure from this initial template.

A philosophically modular structure is the structure corresponding to a philosophically modular theory.

§2 Philosophically modular structures

- Note that philosophically modular structures, should they exist, are not necessarily determinate.
- In particular, they need not be determinate in cardinality.
- But can we find some natural examples of philosophically modular theories/structures?

How can we make this formal?

Definition.

Let $\mathbb{G}$ be a set and $cl : \mathcal{P}(\mathbb{G}) \to \mathcal{P}(\mathbb{G})$ be a function (the closure operation). Then $(\mathbb{G}, cl)$ is a pre-geometry iff:

(i) $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$.

(ii) If $A \subseteq B$ then $cl(A) \subseteq cl(B)$.

(iii) If $a \in cl(A \cup \{b\}) \backslash cl(A)$ then $b \in cl(A \cup \{A\})$.

(iv) If $a \in cl(A)$ then there is a finite $A_0 \subseteq A$ such that $a \in cl(A_0)$.

DEFINITIONS.

If $(\mathbb{G}, cl)$ is a pre-geometry then:

(I) A set $B \subseteq \mathbb{G}$ is independent iff $c \notin cl(B \backslash \{c\})$ for all $c \in B$.

(II) A set $A \subseteq \mathbb{G}$ is closed iff $A = cl(A)$.

(III) A subset B of a closed set A is a basis of A iff B is independent and $cl(B) = A$.

(IV) The dimension of a closed set A is the cardinality of any basis of A.

Important: These definitions are effectively generalisations of what you get with garden-variety spaces like Euclidean space: The dimension tells you how many coordinates are needed to specify a point in the geometry.

§3 Strongly minimal theories and structures

Definitions.

- Let $\mathfrak{M}$ be a model of a countable, complete theory T with universe M (and assume that T has infinite models). Let

$$\phi(\mathfrak{M}) = \{\bar{a} \in M^n | \mathfrak{M} \models \phi(\bar{a})\}$$

  be any infinite definable subset in $\mathfrak{M}$. Then $\phi(\mathfrak{M})$ is minimal in $\mathfrak{M}$ iff for all $\mathscr{L}(\mathfrak{M})$-formulas $\psi(\bar{x})$ the intersection $\phi(\mathfrak{M}) \cap \psi(\mathfrak{M})$ is either finite or cofinite in $\phi(\mathfrak{M})$.
- A formula $\phi(\bar{x})$ is strongly minimal iff $\phi(\bar{x})$ defines a minimal set in every elementary extension $\mathfrak{N}$ of $\mathfrak{M}$ (and we also say that $\phi(\mathfrak{M})$ is strongly minimal in this case).
- Such a theory T is strongly minimal if the formula $x = x$ is strongly minimal.

§3 Strongly minimal theories and structures

- Given a strongly minimal set $\mathbb{G} = \phi(\mathfrak{M})$, it will be defined using parameters from some finite A_0.
- We can then define a closure operation $cl(A) = acl(A \cup A_0) \cap \mathbb{G}$, where $acl(B)$ is the model-theoretic notion of algebraic closure, i.e. the set of elements $c \in M$ s.t. there is a formula $\psi(x)$ with parameters from B such that $\mathfrak{M} \models \psi(c)$ and only finitely many elements of M satisfy $\psi(x)$ in $\mathfrak{M}$.

Fact.

Given these definitions $(\mathbb{G}, cl)$ is a pre-geometry.

§3 Strongly minimal theories and structures

- We've now got a notion of strongly minimal theory and can now ask whether there is an indeterminate structure corresponding to the theory (call such a thing a philosophically strongly minimal structure).
- Such a structure will be indeterminate in Cardinality (but not Truth).
- We can think of such a structure as philosophically modular: The strongly minimal set provides our base structure and the pre-geometry is our instructions for generating new structures given some cardinal base.
- But it's not determinate which base cardinality we pick.

§4 Comparing strongly minimal and determinate structures

- Given the idea of such a structure, should we either interpret it nominalistically or as a fundamentally higher type as compared to determinate structures?
- The structuralist has to hold that a categoricity theorem provides extra ontological 'juice' that just isn't there for a strongly minimal theory.
- It's hard to see what this might be (aside from determinateness, which begs the question).

§4 Comparing strongly minimal and determinate structures

- For many categorical theories, we know almost nothing about their models.
- ZFC_2 (with anti-large cardinal axioms) is a particularly egregious example, but PA_2 ain't so rosy either (do we really understand if/how *Con*(your-favourite-large-cardinal-axiom) is true?).
- Both have generating ideas that (given second-order logic) yield something determinate.
- But how the construction can be controlled is beyond us.
- By contrast a strongly minimal theory/structure is highly controlled.
- The theory is complete (so Truth is determinate) and the construction of a model is completely controlled by the strongly minimal set and pregeometry (just name a number and I'll tell you what the model is).

§4 Comparing strongly minimal and determinate structures

It is useful here to consider what I'll call Baldwin's Objection.

- Back in 1900, there wasn't a neat separation between talk of structures and theories.
- This was then cleared up in the work of Hilbert, Veblen, Skolem, Gödel, Frankel, Bourbaki, Tarski, Robinson.
- "Structure" just means structure-in-the-model-theoretic-sense.
- Nowadays we have a good distinction between model-theoretic structures and classes of model-theoretic structures (e.g. isomorphism classes, homomorphism classes, class of all $\mathfrak{M} \models \mathsf{T}$).
- Talk of "the natural number structure" or "the group structure" or "the first-order structure of the integers" (separate from specific models) is just using the word "structure" in a retrograde way.

§4 Comparing strongly minimal and determinate structures

- This just rejects philosophical structuralism (since we can't be only talking about structure in mathematics).
- But it helps to elucidate the fact that if we're structuralists we're already committed to some indeterminateness, namely indeterminateness in the system we consider.
- So why should strongly minimal theories/structures motivate higher-order (or no) ontological commitment as compared to isomorphism invariant structure?
- (In particular, note that there's no obvious analogue of the Henkin construction for strongly minimal theories.)

§5 Open Problems

- I want to mention a few open problems.
- First: How should we handle philosophically modular structures and other indeterminate structures so that they can be of the same ontological type?
- One suggestion (currently I'm working on this with Tim Button) use a version of the internalist structuralism (as given in [Button and Walsh, 2018])
- Note that there are some things that we do want to say that are higher-order (e.g. the integer structure under addition and the rotations on the equilateral triangle both instantiate the group structure).
- There is a balancing act to be played.

§5 Open Problems

- In the case of determinate structures, we have some substantive accounts of what they are like.
- e.g. [Shapiro, 1997], [Leitgeb, 2020].
- You can think of these as resulting from a kind of abstraction operation on (say) the isomorphism classes.
- But it's very unclear what the analogue might be for indeterminate structures.
- e.g. What's a position in an indeterminate structure?

§5 Open Problems

Next question: Are the philosophically modular theories/structures exactly the strongly minimal ones?

Theorem.

(Implicit in Baldwin-Lachlan Theorem) Suppose that **T** is uncountably categorical (i.e. has one model up to isomorphism in every uncountable κ). Then **T** has a countable model $\mathfrak{M}$ with a strongly minimal set $\mathbb{G}$ such that:

(1.) For any model $\mathfrak{N} \models \mathbf{T}$ there is an elementary embedding $j : \mathfrak{M} \rightarrow \mathfrak{N}$.

(2.) Any model $\mathfrak{N} \models \mathbf{T}$ of uncountable cardinality λ has $dim(\mathbb{G}(\mathfrak{N})) = \lambda$.

(3.) Any models $\mathfrak{N}, \mathfrak{N}'$ of **T** with $dim(\mathbb{G}(\mathfrak{N})) = dim(\mathbb{G}(\mathfrak{N}'))$ are isomorphic.

- So there's a sense in which uncountably categorical theories admit of a kind of philosophical modularity too.

§5 Open Problems

- More generally, there's a large number of distinctions to be made here (e.g. [Morales et al., 2019] argue that the stability hierarchy measures *distance* from uniqueness in some sense).
- Do these also represent a kind of philosophical modularity?

§5 Open Problems

Going the other way, we can consider:

Conjecture.

The Zilber Trichotomy Conjecture (roughly speaking), states that the geometry of every strongly minimal set is either (i) trivial, (ii) vector-space-like (modular), or (iii) field-like (non-modular).

As it turns out the conjecture is false [Hrushovski, 1993] gave an example of a strongly minimal set that did not fit this template. However this raises the following questions:

- How can we classify philosophical modularity within the strongly minimal theories/structures?
- In particular, should we regard the Hrushovski example as giving us a counterexample to philosophical modularity, or a surprising (unforeseen) consequence of modularity?

Conclusions

- We've seen that strongly minimal theories and structures formally exemplify a kind of philosophical modularity in mathematics.
- Their study puts pressure on the idea of the isomorphism invariant (i.e. determinate) structures as fundamental.
- There's a whole raft of questions about the implications of philosophical modularity and model theory to be addressed.
- This shows that even if you don't want to use model theory to provide a foundation for mathematical structuralism, it can still tell you about the kinds of structure out there, and the ways we think about and construct structures.

Thanks

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