Some Systems of Set Theory on Which Every Set is Countable; or Countabilism and Maximality

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10 December 2019

Introduction

In the 1920s we discovered the following theorem:

Theorem.

The Löwenheim-Skolem Theorem. Let **T** be a first-order theory. If **T** has an infinite model, it has a model in every infinite cardinal.

In particular, a first-order set theory that implies the existence of uncountable sets (e.g. **ZFC**) has models that think that they contain uncountable sets, when in fact they are countable.

> *Thus, axiomatizing set theory leads to a relativity of set-theoretic notions, and this relativity is inseparably bound up with every thoroughgoing axiomatization. ([Skolem, 1922], p. 296)*

Introduction

- In this talk we'll argue that there's further support for this view from set-theoretic developments concerning **ZFC** over the last century...
- ...but there's a revision to our axioms that we can make that dispenses with the relativity of set-theoretic notions.

Plan

Target.

There are natural axiomatisations of set theory, motivated about considerations of maximality on which:

(i) Every set is countable.

(ii) The continuum is a proper class.

(iii) We have substantial consistency strength...sort of...

- §1 Forcing and the Cohen-heim-Skolem Paradox.
- §2 A different take: Doubting the Powerset Axiom.
- §3 The Forcing Saturation Axiom
- §4 The Axiom of Set Generic Absoluteness
- §5 The Extreme Inner Model Hypothesis
- §6 The Ordinal Inner Model Hypothesis
- §7 Remarks, Conjectures, and Open Questions

§1 Forcing and the Cohen-heim-Skolem Paradox

- In 1963 Paul Cohen discovered forcing, settling the independence of the Continuum Hypothesis.
- Here, we take a model $\mathfrak{M} = (M, E) \models$ **ZFC**...
- ...and using a special kind of partial order $\mathbb{P}$ and ingenious way of naming sets...
- ...define a 'new' set $G \subset \mathbb{P}$, 'add' it to M, and close under the operations definable in M to form the forcing extension $\mathfrak{M}[G]$.
- Paul Cohen used this to show that there are very bad failures of CH.

Observation 1.

Forcing can push the continuum arbitrarily high.

§1 Forcing and the Cohen-heim-Skolem Paradox

- Forcing has become a standard part of the set-theorist's toolkit (in fact much of set theory now consists in constructing models, rather than toiling away in **ZFC**).
- It can also be used to collapse cardinals:

Theorem.

[Lévy, 1963] Let κ be any cardinal in some $(M, E) \models \mathbf{ZFC}$. Then there is a forcing partial order $Col(\omega, \kappa)$, such that $\mathfrak{M}[G] \models$ "κ is countable".

Observation 2.

Any cardinal can be made countable by forcing.

§1 Forcing and the Cohen-heim-Skolem Paradox

The Cohen-heim-Skolem Paradox.

We think, by Cantor's reasoning, that there are uncountable sets...but according to forcing, I can always 'dream up' a function that collapses any particular cardinal to countable...but isn't the universe supposed to contain all possible sets? Why is it 'missing' the collapsing generics?

§2 A different take: Doubting the Powerset Axiom

Let's return to our two observations:

Observation 1.

Forcing can push the continuum arbitrarily high.

Observation 2.

Any cardinal can be made countable by forcing.

- Normally these two observations are taken to show that there are models that are radically non-standard in some sense; they are countable, or non-well-founded, or Boolean-valued.[1]

[1] See [Barton, 2019] for a survey.

§2 A different take: Doubting the Powerset Axiom

- But what if we just took these results completely at face value?
- Perhaps forcing shows you that the Powerset Axiom is false: Any time you assume that $\mathcal{P}(\omega)$ exists, I can transcend it through forcing.
- Perhaps instead reals can be added unboundedly and the continuum is a proper class.
- This has been speculatively suggested:

Perhaps we would be pushed in the end to say that all sets are countable (and that the continuum is not even a set) when at last all cardinals are absolutely destroyed. But really pleasant axioms have not been produced by me or anyone else, and the suggestion remains speculation. A new idea (or point of view) is needed, and in the meantime all we can do is to study the great variety of models. ([Scott, 1977], p. xv)

§2 A different take: Doubting the Powerset Axiom

- So let's drop the Powerset Axiom.
- From now on we work the following theories: **ZFC**−Powerset, **NBG**−Powerset, and **MK**−Powerset, which we'll denote by $\mathbf{ZFC}^-$, $\mathbf{NBG}^-$, $\mathbf{MK}^-$ respectively.
- We have two immediate closely-linked challenges given this theory:

Democratic Challenge.

How are we able to find representatives for our usual friendly mathematical structures (e.g. $\mathbb{R}$)?

Strength Challenge.

Usually we want set theory to act as a foundation for mathematics not just in the sense of finding representatives for our usual mathematical structures, but also being able to certify that theories are consistent.

§3 The Forcing Saturation Axiom

- We thus have a challenge to incorporate our two observations (that the reals can be shot arbitrarily high and we can collapse arbitrarily many cardinals) with our two challenges.
- Perhaps let's take our initial observations and just assert the existence of generics directly.

Definition.

($\mathbf{ZFC}^-$) The Forcing Saturation Axiom (or FSA). If $\mathbb{P}$ is a forcing poset, and $\mathcal{D}$ is a set-sized family of dense sets, then there is a filter $G \subseteq \mathbb{P}$ intersecting every member of $\mathcal{D}$. The theory of Forcing Saturated Set Theory or **FSST** comprises **ZFC**$-$Powerset$+$FSA.

§3 The Forcing Saturation Axiom

- The FSA thus asserts that for any partial order $\mathbb{P}$ and any set-sized family of dense sets $\mathcal{D}$, there is a $\mathbb{P}$-generic for $\mathcal{D}$.
- We have immediately:

Fact.

The FSA is equivalent over $\mathbf{ZFC}^-$ to the claim that every set is countable.

Corollary.

FSST is consistent relative to $\mathbf{ZFC}^-$ and is consistent with $V = L$.

§4 The Axiom of Set Generic Absoluteness

- The FSA is thus rather weak.
- We thus need a new idea.
- This will be the idea of absoluteness, things that are possible (can be 'dreamed up') are actual.
- Note that this responds to our original complaint from the Cohen-heim-Skolem Paradox.

§4 The Axiom of Set Generic Absoluteness

- Absoluteness characterisations have already been found for various forcing axioms, e.g. MA, BPFA ([Bagaria, 1997], [Bagaria, 2000]).

Definition.

Absolute-MA. We say that V satisfies Absolute-MA iff whenever $V[G]$ is a generic extension of V by a partial order $\mathbb{P}$ with the countable chain condition in V, and $\phi(x)$ is a $\Sigma_1(\mathcal{P}(\omega_1))$ formula (i.e. a first-order formula containing only parameters from $\mathcal{P}(\omega_1)$), if $V[G] \models \exists x \phi(x)$ then there is a y in V such that $\phi(y)$.

What we get out of an absoluteness principle depends on the following dimensions:

(I) What complexity of formula we reflect.
(II) What parameters we are allowed to use.
(III) What extensions we allow (and where is is reflected).

§4 The Axiom of Set Generic Absoluteness

- With unrestricted parameters and complexity, we immediately get a contradiction in **ZFC** (just collapse ω_1).
- However we are working in $\mathbf{ZFC}^-$, so are more free!

Definition.

($\mathbf{ZFC}^-$) We say that V, a model of $\mathbf{ZFC}^-$, satisfies the Axiom of Set-Generic Absoluteness (ASGA) iff whenever $\phi(\vec{a})$ is a sentence in the language of $\mathbf{ZFC}^-$ in the parameters $\vec{a} \in V$, if $\mathbb{P} \in V$ is a forcing partial order, G is V-generic in the sense that it intersects every dense set in V, and $\phi(\vec{a})$ holds in $V[G] \models \mathbf{ZFC}^-$, then $\phi(\vec{a})$ holds in V.

We can then prove the following two facts:

Fact.

$\mathbf{ZFC}^{-}$ + ASGA implies that $V \neq L$.

Fact.

Unfortunately, $\mathbf{ZFC}^{-}$ + ASGA is equiconsistent with $\mathbf{ZFC}^{-}$.

§5 The Extreme Inner Model Hypothesis

- Whilst the ASGA has substantially more consequences that the FSA, it is still weak (in terms of consistency strength).
- But there we only allowed set-forcing extensions.
- What if we allow other kinds of extension?

Definition.

(**MK**) [Friedman, 2006] Let ϕ be a parameter-free first-order sentence. The Inner Model Hypothesis (or IMH) states that if ϕ is true in an inner model of some outer model of V, then ϕ is already true in an inner model of V.

§5 The Extreme Inner Model Hypothesis

- Again, much of discussion of the inner model hypothesis surrounds how to generalise it to the use of parameters, but we are free without powerset:
- From now on we do need a public health warning, we are still checking the results in what follows:

Definition.

($\mathbf{MK}^-$) Let $\phi(\bar{a})$ be a first-order sentence with parameters $\bar{a} \in V$. The Extreme Inner Model Hypothesis or (EIMH) states that if $\phi(\bar{a})$ is true in an inner model $I^{V^*} \models \mathbf{ZFC}^-$ of $(V^*, \in, \mathcal{C}^*) \models \mathbf{MK}^-$ of V, then $\phi(\bar{a})$ is already true in an inner model $I \models \mathbf{ZFC}^-$ of V.

§5 The Extreme Inner Model Hypothesis

- The EIMH clearly extends the ASGA.
- But unfortunately it goes probably too far:

Theorem.

Let the Dependent Choice Scheme (DCS) be the principle: If a definable (class) relation has no terminal nodes, we can make ω-many dependent choices on its basis. Then there is no transitive model of $\mathbf{NBG}^{-} + \text{DCS} + \text{EIMH}$ where we consider extensions satisfying the DCS.

- That's all a bit technical, but the core point is that the EIMH is incompatible with the justification of even very weak choice principles.

§6 The Ordinal Inner Model Hypothesis

- Let's dial back the parameters a little bit.

Definition.

(**MK**$^-$) Let $\phi(\vec{a})$ be a first-order sentence with ordinal parameters $\vec{a}$. The Ordinal Inner Model Hypothesis (or OIMH) states that if $\phi(\vec{a})$ is true in an inner model $I^{V^*} \models \mathbf{ZFC}^-$ of an outer model of $V \models \mathbf{MK}^-$, then $\phi(\vec{a})$ is already true in an inner model $I \models \mathbf{ZFC}^-$ of V.

§6 The Ordinal Inner Model Hypothesis

Work is ongoing on the OIMH, but we have the following two positive results:

Theorem.

MK$^-$ + OIMH is consistent relative to **ZFC**+ "There are at least ω-many Woodin cardinals."

Theorem.

MK$^-$ + OIMH implies that for every n, there is an inner model $I \models$ **ZFC**$^-$+ "ω_n exists".

- That's still not quite what we want.

Conjecture 1.

We can extend the previous result to obtain **ZFC** in an inner model from the OIMH.

Conjecture 2.

Assuming that we can obtain **ZFC** in an inner model, we can modify the techniques of [Friedman, 2006] to obtain many large cardinals in in inner models.

§7 Remarks, Conjectures, and Open Questions

- We actually do have ways of obtaining large cardinals somewhat artificially (e.g. by stating the existence of mice required to build the models).
- Assume then that we have some way of getting large cardinals in inner models.
- There's a sense in which this Countabilist perspective looks upon the **ZFC**-perspective as impoverished.
- We would then also have made significant progress into the challenge of strength.

§7 Remarks, Conjectures, and Open Questions

- The democratic challenge is harder.
- In our framework there are mathematical structures are either countable or proper-class-sized.
- We can have all our countable structures, and the reals are represented as a proper class.
- Moreover the class of all continuous functions of reals is of cardinality $\mathfrak{c}$, and so is codable by a proper class. ([Holmes, 2017])
- But what about the entire function space on the reals?

- The response will depend on what other models we have floating around, but the core strategy is the following:

Strategy.

Find a **ZFC**-models whose reals are 'close enough' to the class of reals in V, and then those models can be used to study V with whatever **ZFC**-resources we want.

Example: If we have PD in V, and an inner model I with ω-many Woodin cardinals, then since PD yields a high degree of completeness about $H(\omega_1)$, we can learn about V by studying $H^I(\omega_1)$.

Conclusions

- In this talk I've argued that there are legitimate perspectives on which every set is countable.
- There's much more still to be done, and several details need filling in.
- But I don't want to repudiate **ZFC**-based set theory.
- But I do think that there's a substantial challenge for foundations raised:

Challenge.

What exactly are we trying to do with set theory? Actually study higher infinities (e.g. $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\omega))))))$? Or just find representatives for reasoning about 'regular' mathematical objects?

Thanks! Discussion!

Hugely grateful to:
VolkswagenStiftung
FWF
Vera Fischer
Victoria Gitman
Joel Hamkins
Sandra Müller
Chris Scambler
Matteo Viale

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