Relativism and Metalogic; or, Are the Natural Numbers Determinate?

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Introduction

- Please ask questions throughout!
- You can find these slides posted under the 'Blog' section of my website (https://neilbarton.net/). Just Google 'Neil Barton philosophy'. Don't Google 'Barton philosophy'; the footballer Joey Barton has started a philosophy course.
- Mathematics is the most secure, most determinate (in the sense of there being right or wrong answers) discipline out there, right?
- Things aren't so simple...

Plan

Target.

There is a coherent conception of mathematics on which some claims about the natural numbers are not determinate...but if you take this route things get weird (possibly unacceptably so).

- §1 Natural number determinateness.
- §2 The "New York School" and indeterminateness.
- §3 Metalogic and being self-undermining.
- §4 The recourse to feasibility.
- §5 Conclusions.

§1 Natural number determinateness

- It's a really common idea in the philosophy of mathematics, that statements about the natural numbers (i.e. 0, 1, 2, 3,...and so on) are determinate.

NND-Assumption

The Natural Number Determinacy Assumption (NND-Assumption) is the claim that every statement in the language of arithmetic is either true or false (and not both).

- This assumption lies behind a lot of discussion of philosophy of mathematics (e.g. in epistemology: "We don't know the truth value of Goldbach's conjecture.").

§1 Natural number determinateness

- Here's an assumption that one might naturally take to lie behind the NND-Assumption.

ID-Assumption

The Isomorphism Determinacy Assumption (ID-Assumption) is the claim that our thought and use of language concerning the natural numbers determines a unique structure up to isomorphism.

- Why think this?
- Well, start with 0...
- ...take the successor (now we've got 1)...
- ...do it again (now we've got 2)...
- ....keep going...
- That's your structure.
- How to find the truth value? Just look at what's true on this structure.

§2 The "New York School" and indeterminateness

- There's a rich tradition of denying the NND-Assumption (at least, in a form in which we normally think about it) by holding that there actually a bound on the size of things we can talk about (this is known as ultrafinitism, famously championed by Ed Nelson).
- I want to go in a different direction, holding that our talk of natural numbers is coherent (i.e. has a model) but is nonetheless indeterminate.

§2 The "New York School" and indeterminateness

- Recently, emerging from the paper of [Hamkins, 2012] (but with earlier roots) a group of mathematicians (especially Joel Hamkins, Victoria Gitman, Gunter Fuchs, and Jonas Reitz) loosely based around New York, have challenged the NND-Assumption by challenging the ID-assumption.

New York Multiversism.

New York Multiversism is the claim that the subject matter of (**ZFC**-based) mathematics is composed of the models of first-order **ZFC**. Ontologically speaking, any universe is as legitimate as any other.

§2 The "New York School" and indeterminateness

- Why hold this? Well, part of the thought is that modern mathematical thinking is algebraic.

"On the multiverse view, if someone constructs a model of set theory, then we simply wonder, "what would it be like to live in that universe?" Jumping inside, we want to regard it as the real universe." ([Hamkins, 2012], p. 437)

"I am not arguing that the universe view is incoherent, but rather, my point is that if one regards all outer models of the universe as merely simulated inside it via complex formalisms, one may miss out on insights that could arise from the simpler philosophical attitude taking them as fully real."
([Hamkins, 2012], p. 426)

§2 The "New York School" and indeterminateness

- Why indeterminateness then?
- Well, it was (one of) Gödel's genius insights that syntactic notions (e.g. formula, proof) can be coded by natural numbers.

Consistency statements

Given a theory **T**, the consistency statement for **T** (often written $Con(\mathbf{T})$) is the (arithmetically formalised) claim that no proof of $0 = 1$ is provable in **T**.

Theorem.

Gödel's Second Incompleteness Theorem. No consistent (well, roughly speaking, really ω-consistent) **T** capable of representing a small fragment of arithmetic (**PRA**) can prove $Con(\mathbf{T})$.

§2 The "New York School" and indeterminateness

Corollary.

If **ZFC** is consistent, then so is $\mathbf{ZFC} + \neg Con(\mathbf{ZFC})$!

- How can a theory be consistent, but nonetheless think that it is inconsistent (e.g. $\mathbf{ZFC} + \neg Con(\mathbf{ZFC})$)?
- Answer: In a non-standard model of arithmetic.
- The proofs are coded by non-standard natural numbers.

§2 The "New York School" and indeterminateness

- The New York Multiversist thus argues that the ID-Assumption is false (we could be talking about a non-standard model) and hence the NND-Assumption is false.
- But wait, what about our earlier argument (take 0, look at its successor, take 1, look at its successor ... etc.)?
- This argument can be encoded (à la Dedekind) in very weak fragments of second-order logic (e.g. ancestral logic, ω-logic) to produce a proof that all natural number structures (in that logic) are isomorphic.

§2 The "New York School" and indeterminateness

- Hamkins foresees this objection:

  "There seems little reason why two different concepts of set need to agree even on the concept of the natural numbers. Although we conventionally describe the natural numbers as 1, 2, 3, . . ., and so on, why are we so confident that this ellipses is meaningful as an absolute characterization? Peanos categoricity proof is a second-order proof that is sensible only in the context of a fixed concept of subsets of $\mathbb{N}$, and so this ellipses carries the baggage of a set-theoretic ontology." ([Hamkins, 2012], p. 427)

- So Hamkins rejects categoricity for the natural numbers, however we don't need anything like full second-order resources for this (the ability to characterise a finite successor-distance is enough).

§2 The "New York School" and indeterminateness

- Nonetheless, indeterminacy for $\mathbb{N}$ would follow from the following assumption:

FOD-Assumption.

The First-Order Determinateness Assumption (FOD-Assumption) is the claim that only those notions that can be determinately characterised using first-order theories are determinate.

- I won't go in to whether Hamkins actually thinks this (there's some places where he implicitly suggests it).
- We are left with the situation where any **ZFC**-model satisfies the Dedekind categoricity theorem for arithmetic, but what arithmetical structure is latched on to depends on the set-theoretic background in which we start.

§3 Metalogic and being self-undermining

- Interlude: Who remembers the classic objection to the following view?

Logical Positivism (roughly).
Only those statements which are verifiable or falsifiable are meaningful.

- The FOD-Assumption faces a similar problem.

§3 Metalogic and being self-undermining

FOD-Assumption

Only those notions that can be determinately characterised using first-order theories are determinate.

- How to characterise a first-order theory?

First-Order Theories.

A theory **T** is first-order iff it is composed solely of:

- Variables: x_0,..,x_n,..., Function terms (inductively defined), Quantifier symbols $\forall$ and $\exists$, Connectives: $\neg$, $\vee$, etc, Equality: $=$, Predicate symbols: P_0,...,P_n,...
- Formulas built inductively using finitely many applications of certain rules.

§3 Metalogic and being self-undermining

- But now we can recall the following:

Theorem.

(Compactness Theorem) Let **T** be a first-order theory. If **T** is finitely satisfiable (i.e. every finite subset of **T** has a model) then it is satisfiable (i.e. **T** has a model).

Corollary.

([Shapiro, 1991]) Any first-order characterisation of the notion of finiteness has an infinite model. (i.e. There is no first-order sentence that holds on all and only the finite structures.)

- This means that the FOD-Assumption, by its own lights, has indeterminate content.

§3 Metalogic and being self-undermining

- One response (suggested in [Barton, 2016]): Accept that the FOD-Assumption is indeterminate by its own lights...
- ...but regard New York Multiversism, not as an ontological proposal, but rather a way of characterising an algebraic notion of set.
- The notion of first-order theory (and hence **ZFC**) is indeterminate, but we take multiversism to be explaining what a universe that takes itself to satisfy its own conception of **ZFC** can see.
- There is no determinate notion of what **ZFC** is, until we jump in to an arbitrary structure.

§3 Metalogic and being self-undermining

- Unfortunately, this can't work. While any structure satisfying enough set theory will have its own conception of finiteness, there is no way to tell absolutely if such a structure satisfies **ZFC**...

Theorem.

[Hamkins and Yang, 2013] Let M be a model of **ZFC** (with some additional technical properties I won't mention here). Then there are elementary extensions $M \prec M_1, M_2$, such that:

(I) $\mathbf{ZFC}^M = \mathbf{ZFC}^{M_1} = \mathbf{ZFC}^{M_2}$.

(II) $V_\delta^{M_1} = V_\delta^{M_2}$.

(III) $M_1 \models$"$V_\delta \models \mathbf{ZFC}$".

(IV) $M_2 \models$"$V_\delta \not\models \mathbf{ZFC}$"

§4 The recourse to feasibility

- Philosophical upshot: Even if we pick a universe V and then agree on what **ZFC** is, we can't say whether or not V satisfies **ZFC** from within V.
- How to respond to this problem?
- We need to kill off a certain level of indeterminateness.
- Can we do this without accepting the NND-Assumption?
- Recall that the New York School (in particular Hamkins) are concerned with the algebraic ways set theorists can be interpreted with respect to what they say and the kinds of marks they make (e.g. written proofs, diagrams).

§4 The recourse to feasibility

The FFOD-Assumption.

The Feasible First-Order Determinateness Assumption (FFOD-Assumption) is the claim that only notions that can be determinately characterised using statements that can be feasibly written down and/or checked in first-order logic are determinate.

- Note that "There is an infinite set" can be expressed by a single sentence of first-order logic.

Feasible New York Multiversism.

Set-theoretic mathematics is concerned with the algebra of structures satisfying feasible fragments of **ZFC**.

§4 The recourse to feasibility

- This dodges the earlier problem, we do have absoluteness of satisfaction for feasible fragments of **ZFC** (in fact standard fragments, but we can't say this determinately).
- Any time a mathematician says something, we could in principle restrict to just the feasible portion of **ZFC** required to express their claim.
- We have a schematic rather than quantificational commitment to accepting feasible fragments of **ZFC**.
- There are a lot of questions here though...

§4 The recourse to feasibility

- **Question 1.** Isn't feasibility something that we should subject to mathematical analysis?
- e.g. The feasible algorithms are those that run in polynomial time (this deserves additional philosophical scrutiny, but let's run with it for now).
- But these often require natural numbers to be expressed...(e.g. polynomial time is if on input n, the algorithm runs in a time eventually bounded by n^k-many steps, for some constant k).
- So is the notion of feasibility not itself natural-number dependent?
- **Response.** Feasibility is what it is. It can change, and maybe different worlds and times have different notions of feasibility. But given an utterance there is a fact of the matter about what fragments are feasible.
- Of course, each background will have its own conception of what these technical notions of feasibility mean. But real-word feasibility is a different (philosophical) notion that resists mathematical analysis.

§4 The recourse to feasibility

- **Question 2.** What about the things that set theorists say about metatheory?
- e.g. "Models of **ZFC** with the same V_δ agree that V_δ satisfies the same standard formulas of set theory."
- This should be interpreted as: Any time you give me a feasible fragment Γ of **ZFC**, when I analyse the behaviour of V_δ as regards Γ, as long as two structures agree on V_δ then they agree on Γ.
- For a start, this is a lot of reinterpretation.
- But also, it doesn't accord with how we actually do mathematics.
- Once I prove something about all standard fragments, I know it holds about all feasible fragments.
- I don't go through each time and make sure for a new fragment presented that the theorem holds.
- New York Multiversism is meant to make things philosophically more simple, this is making things more complex.

§5 Conclusions

- I think we've seen that denying the NND-Assumption, whilst accepting that our usual talk of natural numbers is coherent, is deeply problematic (New York Multiversism was just one test case, but our arguments should generalise). But...
- I still think, accepting the NND-Assumption, that there is a legitimate algebraic concept of set advanced by New York Multiversism.
- This concept suggests certain technical questions as especially interesting (e.g. set-theoretic geology, modal logic of forcing/potentialism).
- The feasibility response can be understood as providing an analysis of what structures correspond to actual inscriptions made by set theorists (and plausibly leads to new multiversist technical questions—e.g. what species of potentialism?).
- But as a philosophical view of mathematics as a whole it distorts too much.

Thanks! Discussion!

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