Large Cardinals and the Iterative Conception of Set

Neil Barton
Kurt Gödel Research Center

14 December 2018

Introduction

- When thinking about this presentation, I tried to tackle one of the most difficult questions facing philosophers (and academics more widely):

Problem.
How to provide a handout, when I'm always fiddling with my slides the night before?

- You can find these slides posted under the 'Blog' section of my website (https://neilbarton.net/). Just Google 'Neil Barton philosophy'. Don't Google 'Barton philosophy'; the footballer Joey Barton has started a philosophy course.

Introduction

- Large cardinals are (rightly) viewed as some of the most interesting axioms of contemporary set theory.
- Indeed, the study of how they relate to consistency, model-building, and determinacy axioms has been one of the real great successes of the last century in set-theoretic mathematics.
- A lot of philosophical attention has been devoted to the justificatory case for large cardinals.
- The idea that the universe of sets should be maximal (or 'rich' or 'generous', or whatever) has sometimes been mobilised in their favour.

Plan

Target.

There are natural interpretations of maximality on which large cardinal axioms are not maximising principles, but nonetheless can fulfil their usual foundational role.

- §1 The iterative conception of set
- §2 Large cardinals and their foundational role
- §3 Varieties of width maximality
- §4 Can width maximality kill large cardinals?
  - The Axiom of Choice and Reinhardt cardinals
  - Inner model hypotheses and inaccessibles
  - The ultimate forcing axiom and ω_1
- §5 The foundational role redux
- §6 Open questions and conclusions

§1 The iterative conception of set

- The iterative conception of set is one familiar to us, on which sets are viewed as constructed in stages, taking all sets possible at successor stages and iterating for as far as possible.
- We thus need:

1. A notion of a set-building operation.
2. A notion of well-founded iteration.

§1 The iterative conception of set

- What is true then depends on:
    1. What gets formed at successor stages.
    2. How far this operation gets iterated.
- A natural response to the question of set-theoretic truth; the set-theoretic universe should be maximal in some sense.

  "We believe that the collection of all ordinals is very 'long' and each power set (of an infinite set) is very 'thick'. Hence any axioms to such effect are in accordance with our intuitive concept." ([Wang, 1984], p. 553)

- We want to come up with axioms that maximise the subsets at successor stages and the length of the iteration.

§2 Large cardinals and their foundational role

- It's frustrating that there's no concrete formal definition of the notion of large cardinal axiom.
- However, they should have the feature that they transcend the consistency strength of previous large cardinals.
- This can be done by apparent brute size (e.g. inaccessible, hyper-inaccessible, Mahlo).
- Or through certain model-building properties (e.g. $0^\sharp$, the relationship between the least strong and least superstrong cardinal).
- In fact it turns out that the natural large cardinals are (at least apparently) linearly ordered by consistency strength.
- This gives us a first desirable use for large cardinals: Provide the indices of strength for any conceivable mathematics.

§2 Large cardinals and their foundational role

- However, large cardinals are also used in the construction of certain models.
- So, for example, we build models of the form $L[E]$ where E is an extender, using the large cardinal properties attaching to E to build the model.

§2 Large cardinals and their foundational role

- Closely linked to model building are the justificatory cases for axioms of definable determinacy.
- These axioms assert that certain games played with natural numbers have winning strategies, and have desirable consequences with respect to (for example) hereditarily countable sets.
- Importantly, they are implied by large cardinal axioms.
- For example, determinacy for projective sets is implied by the existence of infinitely many Woodin cardinals.
- One might then argue: If we justify the large cardinals, so we justify a nice theory.

§2 Large cardinals and their foundational role

- Okay so, large cardinals are:
  1. Our natural indices for consistency strength.
  2. Useful for building models.
  3. Provide a possible justificatory case for axioms of definable determinacy.
- But why should we think that they are correct?

§2 Maximality and large cardinals

"To answer this question [i.e. height maximality], a number of principles have been invoked. The ones that are probably best known are principles telling us, effectively, that the hierarchy goes at least as far as a certain ordinal. These include the Axiom of Infinity and the standard large cardinal axioms..." ([Incurvati,], p.4)

"As with any large cardinal, positing a supercompact can be viewed as a way of assuring that the stages go on and on; for example, below any supercompact cardinal κ there are κ measurable cardinals, and below any measurable cardinal λ, there are λ inaccessible cardinals." ([Maddy, 2011] pp. 125–126)

- That's enough for now. But examples can be multiplied (e.g. [Hauser, 2001], parts of some textbooks e.g. [Drake, 1974]).

§3 Varieties of width maximality

- How might we try to get a handle on there being many power sets?
- This seems challenging: e.g. CH, lots of reals or lots of functions?
- We will consider the following two linked sharpenings of maximality in width:

§3 Varieties of width maximality

Maximality through Absoluteness

Suppose that some appropriate ϕ holds in some appropriate extension of V. Then ϕ already holds in some structure in V.

Intuitive motivation: Anything you could 'dream up' is already realised in some structure within V.

Maximality through Genericity

If you could generically generate a set of a certain kind, then you already have one.

Intuitive motivation: Viewing forcing as a way of generating new subsets, your universe is already saturated under certain kinds of forcing.

§2 Maximality and large cardinals

- Here's how we'll argue in the rest of the talk:
- The iterative conception legislates for forming all possible subsets at successor stages, and then iterating this as far as possible.
- So what if the subset forming operation at successor stages kills large cardinals?
- This isn't too challenging with examples like L, since that looks like a minimality principle.
- But what if we can find width maximality principles that kill large cardinals?

§4 Can width maximality kill large cardinals?

- Consider the Axiom of Choice.
- This is a maximality principle.
- If you've got a set x formed at $V_{\alpha+1}$, then you've got all members of x at latest at V_α, and so all elements of members of x at latest at V_α, and so a Choice set for x gets formed at latest at $V_{\alpha+1}$.
- (There are counterarguments here, but I don't think they pass muster, and you can bolster this argument in various ways (e.g. with second-order logic as in [Potter, 2004].)).
- Moreover, the Axiom of Choice is a species of maximality through genericity.

Theorem.

(Goldblatt, Todorčević) $\forall\kappa \mathsf{FA}_\kappa(<\kappa\text{-closed})$ is equivalent (modulo **ZF**) to the Axiom of Choice.

§4 Can width maximality kill large cardinals?

- Large cardinals are often defined via the critical points of embeddings $j : M \rightarrow N$.
- One way of strengthening large cardinal axioms is increase the closure of M and N, and specify where j sends the ordinals. e.g.

Definition.

A cardinal κ is λ-supercompact iff κ is the critical point of a non-trivial elementary embedding $j : V \longrightarrow M$ for some transitive inner model $\mathfrak{M} = (M, \in)$, $j(\kappa) > \lambda$, and ${}^{\lambda}M \subseteq M$.

- This suggests natural generalisations (studied recently by Woodin and Koellner):

Definition.

A cardinal κ is Reinhardt iff it is the critical point of a non-trivial elementary embedding $j : V \rightarrow V$.

§4 Can width maximality kill large cardinals?

- But now we have the following:

Theorem.

[Kunen, 1971] Assuming **ZFC**, there are no Reinhardt cardinals.

- It's still open though whether there could be Reinhardt cardinals in a model satisfying only **ZF**.
- Moreover, there's an entire choiceless choiceless hierarchy with some nice consistency implications (see [Woodin, 2011] here), that would outstrip the usual hierarchy.

Theorem.

(Woodin) $Con(\mathbf{ZF}_2 +$ "There exists a Reinhardt cardinal)" $\Rightarrow$
$Con(\mathbf{ZFC}+$"There exists a proper class of ω-huge cardinals").

§4 Can width maximality kill large cardinals?

- Should we say Choice is false then?
- NO! The formation of Choice sets (via generic saturation) at successor stages in V prohibits the formation of a stage with a Reinhardt cardinal.
- On the assumption that Reinhardts are consistent with **ZF** and realised in inner models, the action of the axiom asserting the existence of a Reinhardt cardinal is thus to restrict width.

§4 Can width maximality kill large cardinals?

- We can now make this phenomenon more extreme:

Definition.

(**NBG**) Let $\mathfrak{M} = (M, \in, \mathcal{C}^{\mathfrak{M}})$ be a **NBG** structure. The Class-Generic Inner Model Hypothesis (or CIMH) is the claim that if a (first-order, parameter free) sentence ϕ holds in an inner model of a tame class forcing extension $\mathfrak{M}[G] = (M[G], \in, \mathcal{C}^{\mathfrak{M}[G]})$ of $\mathfrak{M} = (M, \in, \mathcal{C}^{\mathfrak{M}})$, then ϕ holds in an inner model of $\mathfrak{M}$.

- We can use this axiom to say that V has been maximised with respect to internal consistency.
- Anything you can dream up in a class-like context using class forcing, is already realised in a class-like context (i.e. it's a form of maximality through absoluteness).

§4 Can width maximality kill large cardinals?

But we now have:

Theorem.

(modified from [Friedman, 2006]) Suppose V satisfies the Class-Generic IMH. Then there are no inaccessibles in V.

However, we also have:

Theorem.

(modified from [Friedman et al., 2008]) Suppose that V satisfies the Class-Generic IMH. Then V contains an inner model with measurable cardinals (of arbitrarily large Mitchell order).

§4 Can width maximality kill large cardinals?

- We can actually make this slightly more problematic for the advocate of large cardinals.
- Consider Maddy's notion of a theory being restrictive:
- Roughly (and skating over some technical details) a theory $\mathbf{T}_2$ maximizes over another $\mathbf{T}_1$ iff $\mathbf{T}_2$ does one of:
  - (I) Proves that there is (definably) an inner model with $\mathbf{T}_1$, or
  - (II) Proves that there is a truncation at an inaccessible with $\mathbf{T}_1$.
  - (III) Proves that there is an inner model of a truncation at an inaccessible with $\mathbf{T}_1$.
- and $\mathbf{T}_2$ has sets outside that interpretation.

§4 Can width maximality kill large cardinals?

- $\mathbf{T}_2$ properly maximizes when $\mathbf{T}_2$ maximizes over $\mathbf{T}_1$ but not vice-versa.
- $\mathbf{T}_2$ inconsistently maximizes over $\mathbf{T}_1$, when $\mathbf{T}_2$ properly maximizes over $\mathbf{T}_1$ and $\mathbf{T}_2$ is inconsistent with $\mathbf{T}_1$.
- $\mathbf{T}_2$ strongly maximizes over $\mathbf{T}_1$ when $\mathbf{T}_2$ inconsistently maximizes over $\mathbf{T}_1$ and there is no consistent extension of $\mathbf{T}_1$ that properly maximizes over $\mathbf{T}_2$.
- $\mathbf{T}_1$ is restrictive iff there is a consistent $\mathbf{T}_2$ that strongly maximizes over $\mathbf{T}_1$.

§4 Can width maximality kill large cardinals?

Fact.

(see [Barton, S]) Let **ZFC**$^{\text{CIMH}}$ be the class of **ZFC**-consequences of **NBG** + CIMH. Then **ZFC**$^{\text{CIMH}}$ strongly maximizes over **T** = **ZFC**+ "There are α-many measurables" for every α.

- Maddy herself acknowledges that her notion isn't perfect, but it at least gives us a precise sense in which we might say that the Class-Generic Inner Model Hypothesis really does capture some maximising features.
- Note: I am couching things in terms of models here, but really we should think of these as syntactic translations. The details are a bit fiddly and are available in the paper.

Some nice prose...

"I see that there are any number of contradictory set theories, all extending the Zermelo-Fraenkel axioms: but the models are all just models of the first-order axioms, and first-order logic is weak. I still feel that it ought to be possible to have strong axioms, which would generate these types of models as submodels of the universe, but where the universe can be thought of as something absolute. Perhaps we would be pushed in the end to say that all sets are countable (and that the continuum is not even a set) when at last all cardinals are absolutely destroyed." ([Scott, 1977], p. xv)

§4 Can width maximality kill large cardinals?

- Okay, let's go really bananas.
- Forcing axioms are naturally understood as maximality principles, asserting that there are generics for certain kinds of well-behaved posets and families of dense sets.
- They can be understood as asserting that the universe has been saturated under forcing of a particular kind.
- In this way we might think of forcing as generating subsets given some subsets you already have (e.g. see [Venturi, 2019]).

§4 Can width maximality kill large cardinals?

So maybe we should have this axiom:

Definition.

We say that V satisfies the Forcing Saturation Axiom (or FSA) iff for any partial order $\mathbb{P} \in V$, and any family of dense sets $\mathcal{D} \in V$, there is a generic G for $\mathbb{P}$ and $\mathcal{D}$ in V.

- Of course, the FSA is inconsistent with **ZFC**!
- But maybe things are more subtle than that.
- Maybe there are so many subsets of an infinite set that they cannot all be collected at an additional stage.
- Maybe there is a different notion of collecting all possible subsets at successor stages.
- Maybe the Powerset Axiom (or indeed "ω_1 exists") is a kind of large cardinal axiom, that can only be true when we leave out subsets from the hierarchy.

§4 Can width maximality kill large cardinals?

- Okay, so drop the Powerset axiom, adopt the FSA, with the continuum now becoming a proper class...

FSST

(for 'Forcing Saturated Set Theory') comprises:

- All axioms of **ZFC** – Powerset.
- (Definable Powerset Axiom) $(\forall x)(\exists y) y = Def(x)$ (where $Def(x)$ is the definable powerset of x).
- FSA
- $\mathbf{FSST}_2$ is the corresponding extension of **NBG**-Powerset.

§4 Can width maximality kill large cardinals?

- How do we define the iterative hierarchy here?
- Initial idea:

The Naive Forcing Saturated Hierarchy

is defined as follows (within **FSST**):

(I) $N_0 = \emptyset$

(II) $N_{\alpha+1} = Def(N_\alpha) \cup \{G | \exists \mathbb{P} \in N_\alpha \exists \mathcal{D} \in N_\alpha$ "$\mathbb{P}$ is a forcing poset $\mathcal{D}$ is a family of dense sets of $\mathbb{P}$ and G intersects every member of $\mathcal{D}$"$\}$

(III) $N_\lambda = \bigcup_{\beta<\lambda} N_\beta$

(IV) $N = \bigcup_{\alpha \in On} F_\alpha$.

Who can spot the problem?

§4 Can width maximality kill large cardinals?

- We need the generics to be fed in slowly and unboundedly.
- We can commit to a restricted form of possibility: You can only every grab at the 'next' generic in line.
- This is codified by a well-order R, and we have:

The Forcing Saturated Hierarchy

is defined as follows (within **FSST** plus a predicate for R):

(I) $F_0 = \emptyset$

(II) $F_{\alpha+1} = Def(F_\alpha) \cup \{G | \exists \mathbb{P} \in F_\alpha \exists \mathcal{D} \in F_\alpha$ "$\mathbb{P}$ is a forcing poset $\mathcal{D}$ is a family of dense sets of $\mathbb{P}$ and G intersects every member of $\mathcal{D} \wedge G$ is the R-least generic for $\mathbb{P}$ and $\mathcal{D}$"$\}$

(III) $F_\lambda = \bigcup_{\beta<\lambda} F_\beta$

(IV) $F = \bigcup_{\alpha \in On} F_\alpha$.

§4 Can width maximality kill large cardinals?

- F clearly satisfies **FSST**.
- It's not quite as neat as **ZFC** and the V_α, since there isn't a guarantee that F contains every set in a model of **FSST**. This is shown by the following:

Fact.

Over **ZFC**$-$Powerset, the FSA is equivalent to the claim "Every set is countable".

§4 Can width maximality kill large cardinals?

- However, we do have the following, if we modify Maddy's definition to consider theories extending **ZFC**$-$Powerset:

Fact.

Let ϕ be a large cardinal axiom such that $L_{\omega_1} \models \phi$ under the existence of $0^\sharp$. Then, **FSST**+ "$0^\sharp$ exists" strongly maximizes over **ZFC** $+ \phi$, and no consistent extension of **ZFC** $+ \phi$ can strongly maximize over **FSST**+ "$0^\sharp$ exists".

§5 The foundational role for large cardinals on these perspectives

- What then of the foundational roles of large cardinals discussed earlier?
- Well, the indexing of consistency strength is unaffected.
- The case for determinacy is in principle unaffected, since the equivalence is actually with the existence of models, e.g.

Theorem.

TFAE:

1. Projective Determinacy (schematically rendered).
2. For every $n < \omega$, there is a fine-structural, countably iterable inner model $\mathfrak{M}$ such that $\mathfrak{M} \models$ "There are n Woodin cardinals".

- As it happens though, some of the principles we have considered do kill PD (e.g. IMH).

§5 The foundational role for large cardinals on these perspectives

- But it's at least open to hold that we may be convinced by PD on the basis of the various structural relationships exhibited in inner model theory, yet hold that large cardinals are killed.
- PD is, for example, perfectly compatible with **FSST**.
- We can also come up with IMH-like principles that kill some large cardinals but allow for PD (for example, just modify the IMH to only allow universes containing a proper class of Woodins).

§5 The foundational role for large cardinals on these perspectives

- For model building, the production of the canonical model is exactly linked to the determinacy axiom, not the large cardinal itself.
- So we can perfectly well have the construction of the canonical model without the literal truth of the large cardinal.
- Even if the determinacy does fail, given an inner model containing the large cardinal (which we often have for the theories discussed here), we can at least have a context in which the 'canonical' construction can be carried out.
- So whether you think these perspectives interfere with the foundational role for large cardinals, is somewhat dependent on exactly what you need/want.

§6 Open Questions and Conclusions

- There's too many open questions regarding this material to list everything, and there's lots of exciting directions for future research.
- Question 1: How close is the relationship between absoluteness principles and generic existence? (Is absoluteness the broader phenomenon?)
- Question 2: Can we get width-principles destroying large cardinals between Reinhardts and inaccessibles?
- Question 3: Are there strong (but consistent!) set theories that imply that every set is countable?
- Question 4: Can we improve the hierarchy for set theories with only countable sets?
- Question 5: How can we generalise formal notions of restrictiveness to second-order set theories?

§6 Open Questions and Conclusions

- So: I've argued that there are perspectives on maximality, combined with ideas about iterativity, on which large cardinal axioms are false yet fulfil their foundational role, but are not maximality principles.
- But I'm not here to suggest we should replace **ZFC**, or even standard large cardinals (though the questions raised merit answers).
- The main point is just that whether or not something actually counts as a maximality principle or not depends on prior commitments you may have about maximality.
- What we need is a careful disambiguation of the kind of maximality being employed.
- This is starting to get done to an extent (e.g. the unification of large cardinals under the philosophical idea of reflection, inner model hypotheses as absoluteness of the universe).
- What we need is a combination of philosophical and mathematical labour, filling out the mathematical details of each position and underlying philosophical conceptions of sets!

Thanks! Discussion!

Hugely grateful to:

FWF

David Fernández-Bretón

Monroe Eskew

Sy Friedman

Luca Incurvati

Michael Potter

Chris Scambler

Matteo Viale

Barton, N. (S).
Large cardinals and the iterative conception of set.
Manuscript under review.

Bell, J. (2011).
Set Theory: Boolean-Valued Models and Independence Proofs.
Oxford University Press.

Benacerraf, P. and Putnam, H., editors (1984).
Philosophy of Mathematics: Selected Readings.
Cambridge University Press.

Drake, F. R. (1974).
Set Theory: An Introduction to Large Cardinals.
North Holland Publishing Co.

Friedman, S.-D. (2006).
Internal consistency and the inner model hypothesis.
Bulletin of Symbolic Logic, 12(4):591–600.

Friedman, S.-D., Welch, P., and Woodin, W. H. (2008).
On the consistency strength of the inner model hypothesis.
The Journal of Symbolic Logic, 73(2):391–400.

Hauser, K. (2001).
Objectivity over Objects: a Case Study in Theory Formation.
Synthse, 128(3).

Incurvati, L.
Maximality principles in set theory.
Forthcoming in *Philosophia Mathematica*.

Kunen, K. (1971).
Elementary embeddings and infinitary combinatorics.
The Journal of Symbolic Logic, 36:407–413.

Maddy, P. (2011).
Defending the Axioms.
Oxford University Press.

Potter, M. (2004).
Set Theory and its Philosophy: A Critical Introduction.
Oxford University Press.

Scott, D. (1977).
Foreword to *boolean-valued models and independence proofs*.
In [Bell, 2011], pages xiii–xviii. Oxford University Press.

Venturi, G. (2019).
Logic and its Philosophy: contemporary trends in Latin America and Spain, chapter On the naturalness of new axioms in set theory, pages 117–142.
Open Court Publishing, Chicago.

Wang, H. (1984).
The concept of set.
In [Benacerraf and Putnam, 1984], pages 530–570. Cambridge University Press.

Woodin, W. H. (2011).
The transfinite universe.
In Baaz, M., editor, Kurt Gödel and the Foundations of Mathematics: Horizons of Truth, page 449. Cambridge University Press.