Forcing and the Universe of Sets: Must we lose insight?

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Introduction

- When thinking about this presentation, I tried to tackle one of the most difficult questions facing philosophers (and academics more widely):

Problem.
How to provide a handout, when I'm always fiddling with my slides the night before?

- You can find these slides (and indeed the latest version of the paper) posted under the 'Blog' section of my website (https://neilbarton.net/). Just Google 'Neil Barton philosophy'.

Introduction

Much discussion in the philosophy of set theory of the following two sorts of view:

Universism

There is a unique, maximal universe of sets containing all the sets. Call this 'V'.

Width Multiversism

There is no maximal universe. In particular, any universe $\mathcal{V}$ can be forced over to yield sets not in $\mathcal{V}$.

This debate has ramifications for how we view set theory. e.g. How algebraic is set theory? Is it more like number theory or group theory, or something in between?

Introduction

- Often the independence phenomenon is invoked in favour of Width Multiversism.
- Forcing is often argued to put pressure on the Universist, and this seems more challenging than the usual independence of consistency sentences.
- Forcing appears to give you a new ways of constructing sets not in V.
- But why should this be problematic? A universist can interpret forcing constructions (e.g. countable transitive models).

Introduction

- In this paper we'll argue:

Claim 1

There's pressure on the Universist to provide as natural an interpretation as possible of forcing, and link it to absolute truth in V.

Claim 2

She can provide pretty natural interpretations, especially with the use of countable transitive models.

Strategy

- §1 Varieties of forcing
  - §1.1 Set forcing
  - §1.2 Class forcing
- §2 Looking at V through the Multiversist lens
  - §2.1 Proving theorems and formulating axioms
  - §2.2 The naturalness constraints
- §3 Available interpretations
  - §3.1 Forcing relations
  - §3.2 Boolean-valued models
  - §3.3 Boolean ultrapowers and quotient structures
  - §3.4 Countable transitive models.
- §4 Conclusions and directions for future research

§1 Varieties of forcing

- For set forcing, start with a universe $\mathcal{V}$...
- ...find a suitable partial order $\mathbb{P} \in \mathcal{V}$...
- ...and use it (via a clever choice of evaluating names) to define a new set G outside $\mathcal{V}$.
- e.g. Flicking the CH switch.

§1 Varieties of forcing

- Class forcing is largely the same, except we start with a universe $\mathcal{V}' = (V', \in, \mathcal{C}^{\mathcal{V}})$, where $\mathcal{C}$ is some conception of classes for $\mathcal{V}$...
- ...find a suitable class partial order $\mathbb{P} \subseteq V$...
- ...and use it to define a new class G outside $\mathcal{C}^{\mathcal{V}}$, from which new sets may be constructed.
- e.g. Easton forcing and the GCH.

§2 Looking at V through the Multiversist lens

- The argument against the Universist is then as follows (very coarse!):
- Take the Universe V.
- So let's force.
- By the nature of the forcing there are then sets outside V.
- But V was meant to be all the sets!

§2 Looking at V through the Multiversist lens

- Wait, wait, wait. We know that we can interpret the forcing without admitting V-generics.
- e.g. Obtain a countable transitive model for any desired finite fragment of **ZFC**, or talk about Boolean-valued models.
- So when we do a relative consistency proof, we can just find an appropriate interpretation, where we know forcing can be interpreted non-vacuously.
- We just use forcing to show number-theoretic facts about what we can prove from what (hence its role in independence), so any old model will do, right? Wrong.

§2 Looking at V through the Multiversist lens

- Forcing isn't just a methodology for proving relative consistency/independence results, it's a tool for constructing new structures from old.
- This is what makes it so good for relative consistency and independence, but there are other ways we can use it.
- In particular, we can use forcing to prove new theorems in **ZFC** and also formulate new axioms extending **ZFC** (and subsequently prove theorems).

§2 Looking at V through the Multiversist lens

- As regards proving theorems in **ZFC**, we have an embarrassment of riches.
- In fact, a whole book's worth ([Todorčević and Farah, 1995]).

Theorem.

[Malliaris and Shelah, 2016] $\mathfrak{p} = \mathfrak{t}$.

- The idea of the proof is to assume that $\mathfrak{p} < \mathfrak{t}$ in V, move to the extension $V[G]$, and see that a particular set would then have to be both empty and non-empty.
- Notice here we are discovering facts about uncountable objects in V.

§2 Looking at V through the Multiversist lens

- How about formulating axioms?
- **Example 1.** Generic embeddings: We might have a non-trivial $j : V \to M \subset V[G]$.
- A generic embedding involves predicating a property of an object internal to V by considering an elementary embedding that lives in an extension.
- They can be defined by either set or class forcing, and the critical points can be accessible but uncountable.

§2 Looking at V through the Multiversist lens

- **Example 2.** Virtual large cardinals.
- A virtual large cardinal property predicates a large cardinal property of an ordinal κ in V by using resources in extensions, for example:

Definition.

[Schindler, 2000] A cardinal κ is remarkable (or virtually supercompact) iff in the $Col(\omega, {<}\kappa)$ forcing extension $V[G]$, for every regular $\lambda > \kappa$ there is a cardinal $\lambda_0 < \kappa$, λ_0 regular in V, and $j : H^V_{\lambda_0} \longrightarrow H^V_\lambda$ such that $crit(j) = \gamma$ and $j(\gamma) = \kappa$.

- Again we are formulating properties of large objects in V by using extensions. Check out [Gitman and Schindler, 2018]!

§2 Looking at V through the Multiversist lens

- **Example 3.** Inner model hypotheses.

Axiom.

$\mathfrak{M}$ satisfies the Set-Generic Inner Model Hypothesis iff whenever a (first-order, parameter free) sentence ϕ holds in an inner model of a set forcing extension $\mathfrak{M}[G] = (M[G], \in, \mathcal{C}^{\mathfrak{M}[G]})$ of $\mathfrak{M} = (M, \in, \mathcal{C}^{\mathfrak{M}})$ (where $M[G]$ consists of the interpretations of set-names in V using G, and $\mathcal{C}^{\mathfrak{M}[G]}$ consists of the interpretations of class-names in $\mathcal{C}^{\mathfrak{M}}$ using G), then ϕ holds in an inner model of $\mathfrak{M}$.

§2 Looking at V through the Multiversist lens

Axiom.

Again, let $\mathfrak{M} = (M, \in, \mathcal{C}^{\mathfrak{M}})$ be a **NBG** structure. The Class-Generic Inner Model Hypothesis is the claim that if a (first-order, parameter free) sentence ϕ holds in an inner model of a tame class forcing extension $\mathfrak{M}[G] = (M[G], \in, \mathcal{C}^{\mathfrak{M}[G]})$ of $\mathfrak{M} = (M, \in, \mathcal{C}^{\mathfrak{M}})$ (where $\mathfrak{M}[G]$ and $\mathcal{C}^{\mathfrak{M}[G]}$ are defined as above), then ϕ holds in an inner model of $\mathfrak{M}$.

Fact.

[B., Eskew, Friedman] Assuming a V_κ elementary in V, there is a model satisfying the Set-Generic Inner Model Hypothesis that violates the Class-Generic Inner Model Hypothesis.

Combined with generic embeddings defined by the stationary tower, this shows that class forcing allows us to do more.

§2 Looking at V through the Multiversist lens

- How might we use these facts to strengthen the Multiversist's challenge?

"...a set theorist with the universe view can insist on an absolute background universe V, regarding all forcing extensions and other models as curious complex simulations within it. (I have personally witnessed the necessary contortions for class forcing.) Such a perspective may be entirely self-consistent, and I am not arguing that the universe view is incoherent, but rather, my point is that if one regards all outer models of the universe as merely simulated inside it via complex formalisms, one may miss out on insights that could arise from the simpler philosophical attitude taking them as fully real."

§2 Looking at V through the Multiversist lens

- We might say that the Universist, while she can interpret forcing, has to explain how her interpretation is sufficiently natural on her view.
- The Width Multiversist clearly has a very natural position here!
- To what extent can we view V as situated in a multiverse?

§2 Looking at V through the Multiversist lens

The Naturalness Constraints:

(1.) (The Facetious Constraint.) The interpretation of 'V' could refer to V, G to an actual generic outside V, and $V[G]$ to a literal extension of V.

(2.) The interpretation of 'V' in the construction could be V itself.

(3.) More minimally, the interpretation of 'V' in the construction could satisfy the same first-order sentences as V.

(4.) One or both of the interpretations of 'V' and '$V[G]$' could be well-founded, and hence admit of an absolute notion of being formed through transfinite iteration of a powerset-like operation.

(5.) The interpretations of 'V' and '$V[G]$' could contain uncountable sets.

(6.) Each of 'V' and '$V[G]$' could contain all the ordinals.

§2 Looking at V through the Multiversist lens

The Naturalness Constraints:

(7.) The structures denoted by each of 'V' and '$V[G]$' could be two-valued.

(8.) The movement between the interpretation of 'V' and '$V[G]$' could be 'transparent', in the sense that whatever is denoted by '$V[G]$' really is obtainable by the usual forcing idea of the addition of a generic to whatever is denoted by 'V'.

(9.) Steps in proofs that use forcing constructions could be interpreted with the minimal amount of change, so additional or different steps do not need to be made to keep the proof in line with the interpretation.

§3 Available interpretations

- **Idea 1.** Forcing relations. Define relations $\Vdash^*$ in V such that:

1. If $\phi_1, ..., \phi_n \vdash \psi$ and $p \Vdash^*_{\mathbb{P}} \phi_i$ for each i, then $p \Vdash^*_{\mathbb{P}} \psi$.
2. $p \Vdash^*_{\mathbb{P}} \phi$ for every axiom of **ZFC**.
3. If $\phi(x_1, ..., x_n)$ is a formula known to be absolute for transitive models, then for every p and all sets $a_1, ..a_n$; $p \Vdash^*_{\mathbb{P}} \phi(\check{a}_1, ..., \check{a}_n)$ iff $\mathbb{1}_{\mathbb{P}} \Vdash^*_{\mathbb{P}} \phi(\check{a}_1, ..., \check{a}_n)$ iff $\phi(a_1, ..., a_n)$ is true in V.

§3 Available interpretations

- We can then interpret a locution such as "V has an extension $V[G]$ such that ϕ" as "There is a $\mathbb{P} \in V$, $p \in \mathbb{P}$, and $\Vdash_{\mathbb{P}}$ such that $p \Vdash_{\mathbb{P}} \phi$", and use the facts about check-names to pull results back to V.
- Problem. This is very unnatural.
- Although 'V' denotes V here, it's very syntactic, there's no consideration of actual embeddings or sets etc.

§3 Available interpretations

Definition.

Let $\mathfrak{M}$ be a model for **ZFC**. Then the Friedman poset (denoted by '$\mathbb{F}^{\mathfrak{M}}$') is a partial order of conditions $p = \langle d_p, e_p, f_p \rangle$ such that:

(I) d_p is a finite subset of ω.

(II) e_p is a binary acyclic relation on d_p.

(III) f_p is an injective function with $dom(f_p) \in \{\emptyset, d_p\}$ and $ran(f_p) \subseteq \mathfrak{M}$.

(IV) If $dom(f_p) = d_p$ and $i, j \in d_p$, then ie_pj iff $f_p(i) \in f_p(j)$.

(V) The ordering on $\mathbb{F}^{\mathfrak{M}}$ is given by:
$p \leq_{\mathbb{F}^{\mathfrak{M}}} q \leftrightarrow d_q \subseteq d_p \wedge e_p \cap (d_q \times d_q) = e_q \wedge f_q \subseteq f_p$.

Theorem.

[HOLY et al., 2016] (attributed to Friedman) $\Vdash^*_{\mathbb{F}}$ is not uniformly definable for $\mathbb{F}$. (But there's some fascinating relations to second-order set theory here! See [Gitman et al., 2017].)

§3 Available interpretations

- **Idea 2.** Boolean-valued models.
- For set-sized partial orders $\mathbb{P}$ we can find a Boolean completion $\mathbb{B}(\mathbb{P})$ (by finding an equivalent separative partial order and adding suprema).
- Next, we form the Boolean-valued model $V^{\mathbb{B}(\mathbb{P})}$
- We can then show that every theorem of **ZFC** gets Boolean-value $1_{\mathbb{B}(\mathbb{P})}$, and can formulate the locution "V has an extension under $\mathbb{P}$ to $V[G]$ such that ϕ" as "ϕ receives Boolean-value greater than $0_{\mathbb{B}(\mathbb{P})}$ in $V^{\mathbb{B}(\mathbb{P})}$", using facts about check-names to pull results back to V.

§3 Available interpretations

- **Problem 1.** We have another problem of scope.
- The usual way to form the Boolean completion is to add a bottom element, and then add in suprema (roughly).
- But in the case of class-sized partial orders, it's not clear that we can do this, since we don't have enough 'room'.
- In fact (see, e.g. [Holy et al., 2018]) a forcing has a class Boolean completion in a model of second-order set theory exactly when all antichains are set-sized (i.e. it satisfies the Ord-Chain Condition).

- **Problem 2.** Again, for the purposes of interpreting many (the bulk of?) forcing constructions, this looks unnatural.
- We are reasoning about how trees behave, or how ordinals are moved, in a two-valued way.
- We do not seem to be reasoning about 'probabilistic' Boolean-valued sets (well, unless $\mathbb{B} = 2$!).

§3 Available interpretations

- **Idea 3.** Use the Naturalist Account of Forcing:

Theorem.

[Hamkins and Seabold, 2012] If V is the universe of set theory and $\mathbb{B}$ is a notion of forcing, then there is in V a definable class model of the theory expressing what it means to be a forcing extension of V. Specifically, in the forcing language with $\in$, constant symbols $\check{x}$ for every $x \in V$, a predicate symbol $\check{V}$ to represent V as a ground model, and a constant symbol $\mathring{G}$, the theory asserts:

(1) The full elementary diagram of V, relativised to the predicate $\check{V}$, using the constant symbols for elements of V.

(2) The assertion that $\check{V}$ is a transitive proper class in the (new) universe.

(3) The assertion that $\mathring{G}$ is a $\check{V}$-generic ultrafilter on $\check{\mathbb{B}}$.

(4) The assertion that the new universe is $\check{V}[\mathring{G}]$, and **ZFC** holds there.

§3 Available interpretations

- It's important here to see the shape of the proof.
- Take a possibly non-generic ultrafilter U on $\mathbb{B}$.
- Define a map j_U (known as the Boolean-ultrapower) into V, to form a structure $\check{V}_U = \{[\tau]_U | [\![\tau \in \check{V}]\!] \in U\}$.
- The quotient structure $V^{\mathbb{B}}/U$ is exactly the forcing extension of $\check{V}_U$ by U over $j_U(\mathbb{B})$ (the generic is $[\mathring{G}]_U$).
- We can then interpret "V has an extension $V[G]$ such that ϕ" as "$\check{V}_U$ has an extension $V^{\mathbb{B}}/U$ such that ϕ", using the properties of the ultrapower to pull back to V.

§3 Available interpretations

- **Problem 1.** Nothing is done about the problem of scope and Boolean-algebras.
- **Problem 2.** The universes involved are often highly non-standard.

Theorem.

[Hamkins and Seabold, 2012] If $\check{V}_U$ is well-founded, then U is countably complete.

Fact.

If κ is the least measurable cardinal, then the Boolean-ultrapower cannot be used to interpret forcings adding sets below V_κ whilst keeping the ultrapower well-founded.

§3 Available interpretations

- You might already think this is bad...
- ...but this can create problems for interpreting set-theoretic reasoning.

Template.

I want to establish that some Δ^1_1-formula ϕ holds in V. I first force a generic G changing the sets above the least measurable κ to form $V[G]$. I then force to add H below κ, forming $V[G][H] \models \phi$. I then infer by the absoluteness of Δ^1_1-formulas for transitive models that $V \models \phi$.

- Whether or not you can interpret this reasoning without adding significant steps depends on whether or not we interpret the construction stepwise or post-hoc.
- Note though, that a combination of the Boolean-ultrapower with Universism yields the suggestion of an especially natural class of forcing constructions—those where we can keep things well-founded.

§3 Available interpretations

- **Idea 4.** Countable transitive models.
- **Idea 4.1** Take a countable transitive model of a finite fragment. No; not similar enough to V.
- **Idea 4.2.** Take a countable transitive model of **ZFC**. No; still might differ from V.
- **Idea 4.3.** Take a countable transitive model $\mathfrak{V} \equiv V$.

§3 Available interpretations

- This looks pretty good; we have all the usual methods of reasoning, and everything is two-valued, well-founded, etc.
- Hamkins is well-aware of this strategy...

§3 Available interpretations

"There are a number of drawbacks, however, to the countable transitive ground model approach to forcing. The first drawback is that it provides an understanding of forcing over only some models of set theory...the question "Is ϕ forceable?" appears sensible only when asked in connection with a countable transitive model M, and this is an impoverishment of the method."

"A second drawback concerns metamathematical issues surrounding the existence of countable transitive models of **ZFC***: the basic problem is that we cannot prove that there are any such models...As a result, this approach to forcing seems to require one to pay a sort of tax just to implement the forcing method, starting with a stronger hypothesis than one ends up with just in order to carry out the argument." ([Hamkins, 2012], p421)*

- Taxation.
- **Response 1.** We can get away with less:

Definition.

Let $\mathscr{L}_{\in,\bar{\mathfrak{V}}}$ be the language $\mathscr{L}_{\in}$ augmented with a single constant symbol $\bar{\mathfrak{V}}$. **ZFC**$^{\bar{\mathfrak{V}}}$ is then a theory in $\mathscr{L}_{\in,\bar{\mathfrak{V}}}$ with the following axioms:

(I) **ZFC**

(II) $\bar{\mathfrak{V}}$ is countable and transitive.

(III) For every ϕ in $\mathscr{L}_{\in}$, $\phi \leftrightarrow \phi^{\bar{\mathfrak{V}}}$ (by Tarski's Theorem, this is an axiom scheme).

ZFC$^{\bar{\mathfrak{V}}}$ is conservative over $\mathscr{L}_{\in}$!

§3 Available interpretations

- Maybe this is unconvincing though (after all we look at things from the outside and say that $\mathfrak{V}$ provides a truth definition for V).
- Let's say that we do think the countable trasitive model strategy constitutes paying a tax.
- We should distinguish between a technical tax, and an ontological or philosophical tax.
- **Response 2.** Need we worry, if we were paying the tax anyway?
- The Universist already accepts that there are things she can say that go beyond the first-order:
- e.g. There is a unique maximal universe of sets that we denote by V, and every sentence of $\mathscr{L}_\in$ is either true or false in V.
- Formalising a truth predicate Tr, and admitting it's use in the Replacement and Comprehension Scheme, quickly yields a countable transitive model elementarily equivalent to V.

§3 Available interpretations

- So Hamkins claims there's a problem of scope.
- But look, this isn't a barrier to providing an interpretation of a particular forcing, it's a barrier to considering forcing over every model.
- **Response 1.** This is just too much to ask (it's essentially insisting on the Facetious Constraint).
- We have provided an explanation of how any forcing construction is linked to truth in V.
- **Response 2.** We certainly do not argue that you have to interpret forcing by the countable transitive model approach!
- Sometimes the other strategies may be better.
- But there are still some problems of naturalness!
  1. What about uncountable sets?
  2. What about all the ordinals?

Proposition.

[B.] Assume that there is a proper class of measurable cardinals and that we have a $V_\delta \prec V$. Let $\mathfrak{M}$ be a countable elementary submodel of V_δ, $\mathbb{P} \in \mathfrak{M}$ be a set forcing partial order such that $p \Vdash_{\mathbb{P}} \phi$ in $\mathfrak{M}$. Let π be the collapsing function to $\mathfrak{V} \equiv \mathfrak{M} \prec V_\delta \prec V$. Then there is a model $\mathfrak{V}^*$ such that:

(I) $\mathfrak{V}^*$ contains uncountable sets.

(II) $\mathfrak{V}^*$ is transitive.

(III) $\mathfrak{V}^*$ is elementarily equivalent to V.

(IV) There is a $G \in V$ such that G is $\mathfrak{V}^*$-generic for $\pi(\mathbb{P})$ and $\mathfrak{V}^*[G] \models \phi$.

PROPOSITION.

[B. (well, the proof is very indebted to S. Friedman!)] Assume that there is a proper class of measurable cardinals and that we have a $V_\delta \prec V$. Let $\mathfrak{M}$ be a countable elementary submodel of V_δ, $\mathbb{P} \in \mathfrak{M}$ be a set forcing partial order such that $p \Vdash_{\mathbb{P}} \phi$ in $\mathfrak{M}$. Let π be the collapsing function to $\mathfrak{V} \equiv \mathfrak{M} \prec V_\delta \prec V$. Then there is a model $\mathfrak{V}^*$ such that:

(I) $\mathfrak{V}^*$ contains uncountable sets.

(II) $\mathfrak{V}^*$ is transitive.

(III) $\mathfrak{V}^*$ is elementarily equivalent to V.

(IV) There is a $G \in V$ such that G is $\mathfrak{V}^*$-generic for $\pi(\mathbb{P})$ and $\mathfrak{V}^*[G] \models \phi$.

§4 Conclusions and directions for future research and unabashed plugs

- I think we've seen that the Universist can provide a reasonably natural interpretation of different kinds of forcing construction via the use of countable transitive models.
- It's foundationally interesting for the Universist that certain forcing constructions can be interpreted especially naturally (e.g. when the Boolean ultrapower can be well-founded).
- **Question 1.** what about class forcing over uncountable transitive models?
- **Question 2.** Other kinds of extension? See [Antos et al., S]!

§4 Conclusions and directions for future research and unabashed plugs

- I want to close with some remarks about the status of the debate.
- Do I think that the Universist is correct?
- To be honest, I'm not sure the question has an answer.
- [Barton, 2016] argues that we should understand Hamkins as advocating an algebraic framework and distinct set concept (rather than telling us about the ontology of set theory).
- This contrasts with the Universist position that tries to refine our concept of set to yield a solution to the continuum problem.
- I'm not sure then that the two positions are really contradicting one another, except in a very superficial sense—proponents are rather using the language of set theory in very different ways.
- But all this requires much more working out...
- Stay tuned!

Thanks! Discussion!

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