Set Theory and Structures

Neil Barton
Kurt Gödel Research Center
Joint work with Sy-David Friedman
Get the slides at https://neilbarton.net/blog/

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Introduction

- There's been a lot of discussion (often quite impassioned) about the relative merits (and demerits) of set-theoretic and category-theoretic foundations.
- One such issue concerns how we treat mathematical structures on such conceptions of foundations.
- We'll leave it open what structure is (since accounts of structure are contestable).
- Whatever mathematical structures are, it seems that the criterion for structural identity is isomorphism (but again, how to understand isomorphism is controversial, as we'll see).
- A central criticism by category theorists (e.g. [Mac Lane, 1986], [Awodey, 1996], examples can be multiplied) of (material) set theory is that it doesn't respect structure.

Aims

We'll argue for the following two main claims:

Pluralism

We should be inclined towards a methodological pluralism concerning choice of foundational theory.

Structures

(Material) set theory can be a useful theory and language for talking about structures (specifically how our mathematical theories interact with their structures).

Structure (of the talk)

1. §1 Two perspectives: Material and Categorial.
2. §2 Criticisms and Methodological Pluralism.
3. §3 How set theory helps the structural perspective.
4. §4 Conclusions and open problems.

- Feel free to ask questions throughout!

§1 Two perspectives: Material and Categorial

- We'll start with the perpective that's more familiar to philosophers (historically at least).
- In material set theory we have an ontology of sets and a primitive relation $\in$, a list of axioms (often spiced-up **ZFC**), and use the theory to encode mathematical objects.
- **Example 1.** For sets A and B, $A \times B = \{\langle a, b\rangle | a \in A \wedge b \in B\}$.
- **Example 2.** For groups $G = (D_G, *_G)$ and $H = (D_H, *_H)$, $G \times_{\text{Group}} H = (D_G \times D_H, *_{G \times H})$,where $*_{G \times H}$ is defined component-wise for $g \in G$ and $h \in H$:
  $\langle g_1, h_1\rangle *_{G \times H} \langle g_2, h_2\rangle =_{df} \langle g_1 *_G g_2, h_1 *_H h_2\rangle$

§1 Two perspectives: Material and Categorial

- This contrasts with the categorial perspective on which we take mathematics to be encoded by properties corresponding to arrows.
- We have arrows f, g, etc. and primitive relations $Dom(f)$, $Cod(f)$, and $\circ$.
- We can then lay down conditions on what it is to be a category and topos (of various kinds).
- e.g. A product is a system of arrows $Pr_A : A \times B \to A$ and $Pr_B : A \times B \to B$, such that for any pair of arrows $f : C \to A$ and $g : C \to B$, there is exactly one arrow $\langle f, g \rangle : C \to A \times B$, such that the whole thing commutes.
- One doesn't need a new definition between two different categories (e.g. **Set** and **Grp**).

§2 Criticisms and Methodological Pluralism

- We'll start with a criticism of the categorial perspective.
- There's lots to be said here (see our paper, or [Maddy, 2017]), but we'll pick the key criticisms for issues concerning structures.
- One core criticism of category-theoretic foundations comes from Geoffrey Hellman:

§2 Criticisms and Methodological Pluralism

"...this theory [i.e. category theory] itself is presented algebraically, via first-order 'axioms' only in the sense of defining conditions telling us what a category is, together with further ones defining topoi of various sorts. As such these 'axioms' are like the conditions defining a group, a ring, a module, a field, etc. By themselves they assert nothing. They merely tell us what it is for something to be a structure of a certain kind."
([Hellman, 2006], p. 134)

§2 Criticisms and Methodological Pluralism

- Hellman's point is that a foundational theory should assert that certain objects exist, but category theory doesn't do this (in contrast to material set theory).
- **Response 1.** ([Mclarty, 2004]) No-one has ever proposed the axioms of category theory as a foundation, the proposal is rather to assert that some topos or other exists and mathematics either can or should be interpreted there.
- **Response 2.** Irrespective of whether we can modify categorial foundations to yield existential content, it is an algebraic enterprise.
- Litmus test: Do we have a particular structure in mind when we consider some topos or other (e.g. **Set**)? Could we shoot for a categoricity proof? Are 'non-standard' models acceptable interpretations?

§2 Criticisms and Methodological Pluralism

- Category theory is thus the appropriate language for discussing schematic types (i.e. systems of relationships that can be instantiated in many non-isomorphic contexts).
- A good example of a particular kind of schematic type; first-order theory.
- It's very interesting that we can interpret mathematics as concerned solely with schematic types as systematised in category theory.

§2 Criticisms and Methodological Pluralism

- Let's turn to a criticism of material set theory.
- Again there's lots here! (Read the paper!)
- The core problem we'll look at is that set theory fails to respect isomorphism invariance (and hence mathematical structure).
- This problem has been pressed by various authors (probably the most famous is [Benacerraf, 1965]).
- Which sets are to be identified with which mathematical objects? Is two $\{\{\emptyset\}\}$ or $\{\emptyset, \{\emptyset\}\}$? Is $\langle a, b\rangle$ to be identified with $\{\{a\}, \{a, b\}\}$ or $\{\{a, 1\}, \{b, 2\}\}$?
- This results in junk theorems e.g. $5 \in 7$.

§2 Criticisms and Methodological Pluralism

- **Response 1.** This kind of set-theoretic reductionism is extreme.
- We don't have to identify mathematical objects with sets.
- Set theory provides use with a (more maybe several) universe(s) in which we can encode and represent mathematics, allowing us to pinpoint the logical strength of (possibly new and controversial) mathematical theories.
- Finding a model in the sets acts as a certificate that the original (and probably more fluid) mathematical language is in good working order (up to a certain degree of confidence).

§2 Criticisms and Methodological Pluralism

- We thus arrive at the following position:

Methodological Pluralism.

Material set theory is the appropriate language for explaining what can be consistently built (up to a certain level of confidence), and category theory is the appropriate language for relating different schematic types to one another.

§2 Criticisms and Methodological Pluralism

- There's an additional level of subtlety here though: The complaint of non-isomorphism invariance by the category theorist is subject to a tu quoque.
- [Tsementzis, 2016] Lots of statements in the language of category theory are not isomorphism invariant!
- e.g. "f has co-domain B".
- (Side note: This might be avoided by some other systems, e.g. Makkai's **FOLDS** or HoTT.)

- What we do have in many topoi is the following kind of result:

Theorem.

(**ETCS**) Let $\phi(X)$ be a formula in the language of **ETCS** with no constants and no free variables except the set variable X. Then if X and Y are isomorphic, then $\phi(X)$ iff $\phi(Y)$.[a]

[a] See here [McLarty, 1993], p. 495.

- This result yields a way of 'modding out' the non-isomorphism invariant 'noise' for a certain class of formulas.
- When working in set theory, are we stuck with the 'noise' and the limited role we have identified, or can we do more with structures using set theory?

§3 How set theory helps the structural perspective

Idea: Build set theory over an antecedently given theory of structure (about which we remain agnostic) and use the combinatorial power of set theory to prove results about structures and theories, whilst being able to mod out the noise.

1. Set theory and structures (**ZFCS**).
2. Class theory with structures (**NBGS**).
3. Morley Categoricity.
4. Isomorphism types.

§3 How set theory helps the structural perspective

The theory of Set Theory with Structures (or **ZFCS**) is defined as follows:

- Symbols:
  - (A) We have three sorts of variables: $u_0, u_1, ..., u_n, ...$ will range over urelements (to be featureless points in the domains of structures), $s_0, s_1, ..., s_n, ...$ will range over structures, and $x_0, x_1, ..., x_n, ...$ will range over sets.
  - (B) The usual logical symbols (so one's favourite connectives and quantfier(s)), and one non-logical symbol '$\in$' (to denote material set membership).
  - (C) Symbols: A single symbol U (for universes), $f_{m,n}$ (for m-ary functions), $R_{m,n}$ (for m-ary relations) and c_n (for constants), where m, n are natural numbers and $m > 0$. These will be used to describe structures.

§3 How set theory helps the structural perspective

(A) $a = b$ where a, b are variables of the same sort.
(B) $a \in b$ where a is a variable and b is a set-variable.
(C) $U(s, a)$ where s is a structure-variable and a is an urelement-variable. (Intended meaning: a belongs to the universe (or domain) of the structure s.)
(D) $f_{m,n}(s, u_1, ..., u_m) = u$ where s is a structure-variable, the u_i and u are urelement variables. (Intended meaning: the u_i and u belong to the universe of the structure s and the interpretation of the m-ary function symbol $f_{m,n}$ in s sends $(u_1, ..., u_m)$ to u.)
(E) $R_{m,n}(s, u_1, ..., u_m)$ where s is a structure-variable and the u_i are urelement variables. (Intended meaning: The m-tuple $(u_1, ..., u_m)$ belongs to the interpretation of the m-ary predicate symbol by the structure s.)
(F) $c_n(s) = u$ where s is a structure-variable and u is an urelement variable. (Intended meaning: The interpretation of the constant symbol c_n by s is u.)
(G) Compound formulas: Obtained from atomic formulas by closing under connectives and quantifiers in the usual way. (Though, since the language is 3-sorted, there will be three kinds of quantifier.)

§3 How set theory helps the structural perspective

- Axioms:
  - (A) Extensionality for sets.
  - (B) Formula-Foundation for Sets: If a formula holds of some set then it holds of some set which is disjoint from all other sets for which the formula holds.
  - (C) The Axiom of Infinity: Usually rendered as concerning the existence of an inductive pure set.
  - (D) Pairing, Union, Powerset, Separation and Collection for sets.
  - (E) Axiom of Choice for sets.
  - (F) The domain of every structure is a set: i.e. $\forall s \exists x \forall a (U(s,a) \leftrightarrow a \in x)$.
  - (G) The Anti-Urelement Set Axiom: No set contains all of the urelements.

§3 How set theory helps the structural perspective

We can then define **NBGS**:

- Symbols: All the symbols of **ZFCS**, with an additional kind of variables $X_0, X_1, ..., X_n, ...$ for classes.
- Atomic formulas: In addition to the well-formed formulas of **ZFCS**, we admit $X_n = X_m$ for class variables X_n and X_m as well-formed, as well as $v_0 \in X_n$ for class-variable X_n and v_0 is either a set, structure, or urelement variable.
- Compound formulas: Obtained inductively from the connectives, $\in$, urelemente quantifiers, structure quantifiers, set quantifiers, and class quantifiers.

§3 How set theory helps the structural perspective

- Axions:

(A) All axioms of **ZFCS**.
(B) Extensionality for classes (i.e. X_n and X_m are equal iff they have the same members).
(C) Predicative Class Comprehension:

$$\exists X \forall u \forall s \forall x [(\phi(u) \leftrightarrow u \in X) \wedge (\psi(s) \leftrightarrow s \in X) \wedge (\chi(x) \leftrightarrow x \in X)]$$

(Where u is a urelement variable, s is a structure variable, and x is a set variable, there are no class quantifiers in ϕ, ψ, and χ, and each of ϕ, ψ, and χ is free for u, s, and x respectively.)

§3 How set theory helps the structural perspective

We can then have two different kinds of definition of isomorphism:

Structural isomorphism

Two structures s_0 and s_1 are structure-theoretically isomorphic iff there is a third structure s within which there is a binary relation between the universes of s_0 and s_1 satisfying the usual rules of isomorphism.

Set-theoretic isomorphism

Two structures s_0 and s_1 are set-theoretically isomorphic iff there is a set-theoretic bijection between the domains if s_0 and s_1 satisfying the usual rules of isomorphism.

§3 How set theory helps the structural perspective

- One worry: Can the structures see the required isomorphisms?

The Structural Richness Axiom

Any set-theoretic isomorphism has a corresponding extensionally equivalent structure-theoretic isomorphism. In fuller formalism: If f is a set-theoretic isomorphism between s_0 and s_1, then there is an s such that s maps u_α to u'_α iff f does.

The Structural Radical Richness Axiom

Any set-theoretic structure is mirrored by a corresponding structure-theoretic structure. In fuller formalism: For any set X of urelements and set-theoretic functions $f^X_{m,n}$, relations $R^X_{m,n}$ on X, and constants c^X_n in X, there is an s such that $U(s, u_\alpha)$ (for each $u_\alpha \in X$), and structural relations $f^s_{m,n}$, $R^s_{m,n}$, and c^s_n equivalent to $f^X_{m,n}$, $R^X_{m,n}$, and c^X_n in the obvious way.

§3 How set theory helps the structural perspective

Lemma.

[B., Friedman] (**NBGS**) **The Hot Toast Lemma.** Suppose that $\phi(v)$ is a formula without parameters in the language of **NBGS** and v is a variable ranging over structures. Suppose that $\mathfrak{M}$ is a model for **NBGS** and s_0 and s_1 are structures in $\mathfrak{M}$ which are isomorphic in $\mathfrak{M}$. Then $\mathfrak{M} \models \phi(s_0) \leftrightarrow \phi(s_1)$.

Proof.

Look at the paper.(You can just define an automorphism inductively by swapping urelements.) □

§3 How set theory helps the structural perspective

How does this yield interesting information about structures?

Theorem.

[Morley, 1965] Suppose that a countable first-order theory **T** has exactly one model up to isomorphism in a single uncountable cardinal. Then it has one model (up to isomorphism) in every uncountable cardinal.

- But it's trivial to recast this as a claim directly about structures...
- ...and we have all the usual resources around to prove this using the normal set-theoretic apparatus...
- ...and we then know that whatever we prove in the set theory will be true of the structures (by the Hot Toast Lemma)...
- ...so the material set-theoretic perspective can give us information about how our theories (kinds of schematic type!) interact with structures.

§3 How set theory helps the structural perspective

We can also use **NBGS** to select less arbitrary representatives for isomorphism types:

Isomorphism Types

In **NBGS**, we say that a class X consisting of structures is (set- or structure-theoretically) invariant if X is closed under isomorphism between structures. If in addition any two structures in X are isomorphic we refer to X as a (set- or structure-theoretic) isomorphism type.

- This contrasts with the selection of a single set-theoretic object in **ZFC**.

§4 Conclusions and open questions

- There's lots of open questions here! (If you're interested, read the paper!) A selection:
  1. What is the status of the axioms of **NBGS**?
  2. What about **MKS**?
  3. To what extent should cardinality be viewed as a structural property?
  4. Do **NBGS**-like constructions provide us with new kinds of category?
  5. What about non-concrete categories (e.g. the homotopy category [Freyd, 1970])?
  6. Is there any additional insight provided on how category theory and set theory might be blended?

Conclusions

- Whilst we disagree strongly with the idea that set theory is somehow defective for not respecting isomorphism invariance in non-set-theoretic language, it highlights an important line of inquiry:
- Figure out how different foundational perspectives can inform one another.
- Here, we hope to have made a small step in this direction.

Thanks! Discussion! Paper: http://philsci-archive.pitt.edu/14633/

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