Mathematical Gettier Cases

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Introduction

- When thinking about this presentation, I tried to tackle one of the most difficult questions facing philosophers (and academics more widely):

Problem.
How to provide a handout, when I'm always fiddling with my slides the night before?

- You can find these slides posted under the 'Blog' section of my website (https://neilbarton.net/). Just Google 'Neil Barton philosophy'.

Introduction

- We'll start with the following coarse idea, one that might initially seem attractive.

The Logician's Dogma

The standard of mathematical justification is the possession of a proof from axioms for the relevant mathematical subject matter.

- This seems initially attractive because it seems to accord well with mathematical practice (e.g. journal review procedure, content of talks etc.).

Introduction

- It's then hard to see how we could ever have a mathematical Gettier case.
- The standard for mathematical justification is possession of a proof from the axioms...
- But the axioms are true and the rules of logic preserve truth...
- So how could one ever have a true but epistemically lucky justified belief?
- Possession of a proof from the axioms already entails that we're not lucky!
- Even more acute when we consider the usual templates for Gettier cases (e.g. Zagzebski).

Strategy

Target.

The possibility (and in some cases actuality) of mathematical Gettier cases indicates an important role for mathematical explanation, with several upshots for the practice of mathematics.

- §1 Developing the Logician's Dogma
- §2 Black box lemmas
- §3 Axiom selection
- §4 Computer-assisted proof
- §5 Morals
  - §5.1 Jenkins and explanation
  - §5.2 Folklore theorems
  - §5.3 Re-proving theorems
  - §5.4 Methodological pluralism in foundations

§1 Developing the Logician's Dogma

- Okay, there's a sense in which the Logician's Dogma is imprecise...
- What exactly does possession of a proof come down to?
- There are certainly interpretations that allow for easy Gettiering.
- e.g. Ability to produce (possibly on command) something that looks like a proof in first-order logic.
- Easy Gettier cases can then be generated, e.g. the mystic, the spilt ink, the Putnam-Scambler Proof-Theoretic Ant.

§1 Developing the Logician's Dogma

- The problem in these cases seems to be that the agent's understanding and the production of the proof aren't appropriately connected.
- I don't have a proof by simply carrying a copy of one in my back pocket.
- In order to have a proof, one should at least:
  - Understand the concepts involved in the proof.
  - Be able to describe how the steps of the proof are inferred from one another.

§2 Black box lemmas

- An issue here concerns how much one should have understood of the proof one provides.
- Mathematics is replete with the use of lemmas that are often used as 'black boxes'.
- This is an important practice for allowing mathematics to move forward as a community.
  1. It's unreasonable to expect mathematicians to follow through every lemma they rely on.
  2. Using lemmas from subject matters with which one is not familiar is very fruitful.

§2 Black box lemmas

- This facilitates a Gettier case.
- Suppose I use a published, well-peer-reviewed lemma from a different field as a black box in a proof.
- As it turns out, the proof of that lemma is flawed (but the lemma is nonetheless true).
- It seems then that I do have justified true belief—I can perfectly well understand all the elements of my proof (even if I don't understand the proof of the lemma).
- But I don't know, it could easily have been the case that I had ended up refraining from believing my theorem or believing its negation.

§2 Black box lemmas

- One response might be that I fail to satisfy the Logician's Dogma, since I should understand all the elements of the used lemma too.
- But this seems unreasonable and too strict.
- In particular it doesn't seem like I've done anything epistemically culpable here.
- (Upshot) This highlights that often mathematics is epistemically messy with the full details of proofs not necessarily held by any one person.
- There are actual examples of the flavour I describe (e.g. Dehn's Lemma was thought proved in 1910, a flaw was found in 1929, and it was finally proved only in 1957, the Four Colour Theorem was thought proved 1879–1891, examples can be multiplied).
- In fact, this phenomenon is surprisingly common.

§3 Axiom selection

- In that last example, there was at least something defective in the overall proof architecture.
- It's been known for quite some time (at least since [Maddy, 1988]) that selecting axioms for certain subject matters can be tricky.
- Set theory is a good example here, since we have many natural ways we might extend our usual set theory (**ZFC**).
- To take two examples, there are justifications for both the Proper Forcing Axiom and $V =$Ultimate-L (it doesn't matter what these say for the philosophical point).
- These two theories agree on some points (e.g. whether a weakly compact cardinal is consistent) but disagree on others (e.g. the Continuum Hypothesis).

§3 Axiom selection

- Suppose then that I believe PFA in virtue of these good justifications.
- I then prove some ϕ on this basis.
- But as it turns out, $V =$Ultimate-L is true. (Note here: There's some very heavy-duty realism driving this example.)
- Suppose further that $V =$Ultimate-L agrees on ϕ.
- It looks like I have justified true belief in ϕ, but the justification was lucky.
- My false but well-justified lemma has lead me to truth.

§3 Axiom selection

- Again though, it doesn't seem like I've necessarily done anything epistemically blameworthy here.
- There's a whole gamut of different reasons why I might strongly believe PFA.
- Maybe I've been brought up in a mathematical culture that has only exposed me to one bunch of justifications.
- Upshot: At least as far as the practice of mathematics goes, there's not just one tradition.

§4 Computer-assisted proof

- In that last example, we had an obvious false lemma, or at least something defective about how the agent's concepts interacted with reality.
- But I think we can do better...
- Earlier, I mentioned the Four Colour Theorem.
- The original proof of this result is obtained by showing that any (smallest) counter-example must contain one of 1'936 maps.
- These 1'936 maps were then checked by computer and found to be four-colourable, establishing the theorem.

§4 Computer-assisted proof

- We can use the reliance on external apparatus to Gettier...
- Suppose that the verification has been run numerous times, and everyone understands the concepts and software.
- But also assume, by grand cosmic chance, that in every simulation there was a hardware failure, subsequently corrected by a different hardware failure.
- However, all this goes on at the level of hardware, the actual output may be in good working order.
- It seems reasonable to say that we don't know here; our knowledge is based on a successful simulation, but we don't have one!
- But we also seem justified—it seems rather strong to say we aren't!

§4 Computer-assisted proof

- There was some controversy surrounding the four-colour theorem, but there are other areas of (pure!) mathematics where computer-verification is completely standard (e.g. classification of simple finite groups).
- More generally, our own thought is related to computational processes.
- We need the world underwrite the formation of proof in the right way (e.g. the expert brain surgeon and a hammer).
- Upshot: Mathematicians are limited beings in a messy world, and our account of mathematical knowledge has to tolerate this.

§5 Morals

- Gettier-cases in mathematics have appeared in a couple of places in the literature (e.g. [Jenkins, 2008]).
- For Jenkins, S knows that P iff:
  - S believes P.
  - P is true.
  - P is a good explanation for S believing that P, for someone not acquainted with the particular details of S's situation (an 'outsider').

§5 Morals

- This avoids all my Gettier-cases, since the explanatory link is undercut.
  - In the case of the black box there's an omission.
  - In the case of axiom selection, there's an omission.
  - For computer-assisted proof, the explanatory link is genuinely severed.
- Jenkins, however, only (to my knowledge) considers the more 'easily obtainable' kind of Gettier cases (her examples are roughly analogous to the mystic and false testimony).
- What I think the examples I've given show is that the explanatory link might be totally epistemically inaccessible to us (perhaps even in principle!).

§5 Morals

- I think this has some concrete payoffs for how we go about doing mathematics.
- We need to do all we can in mathematics to ensure that the relevant explanatory links are in place.
- 1. Folklore theorems are often bad.
- An example of Rittberg, Tanswell, and Van Bendegem: A young topos theorist (Olivia Caramello) had extreme trouble trying to publish her 'duality theorem' in topos theory, for the reason it was 'folkloric'.

§5 Morals

"Although considered "folkloric" by some experts, the result does not appear in the literature. I had believed that one could directly deduce it from the theory of classifying toposes of Makkai and Reyes. It is only recently, in the context of a discussion with Caramello, Johnstone and Lafforgue, that the latter attracted my attention to an aspect of Caramellos proof which I had missed... Surprised by this observation, I tried to exhibit the "folkloric" proof that I thought I had of this theorem. With my great astonishment, it took me a night of work to construct a proof based on my knowledge of the subject, and the proof depended only partially on Makkai-Reyes theory!... I draw from this experience the following conclusions: (1) that I had misread Caramellos paper; (2) that the duality theorem may falsely appear straightforward; (3) that the result is non-trivial; (4) that the result is original, since it does not appear in the literature." (André Joyal, in a public letter to Olivia Caramello)

§5 Morals

- Note that a substantial part of Caramello's result, even if you think that the result is relatively easy, was in clarifying the underlying logical (and hence presumably explanatorially informative) links.
- It seems then with folklore theorems the following dichotomy obtains:
  1. Either the proof really is just a tedious and/or unilluminating exercise, and so it can be flagged as such (possibly with a hint on how it should go).
  2. Or, it's at least somewhat non-trivial, in which case it's worth having it in the literature for inspection and drawing out links.

§5 Morals

- 2. Re-proving theorems with different proofs is really important.
- This is a common and accepted practice in mathematics.
- The mathematical usefulness of this practice is clear.
- However, the examples showcase that there's substantial epistemic payoff too: Unfolding the explanatory links in different areas.

§5 Morals

- 3. We should be inclined towards a methodological pluralism concerning mathematics (including foundations!).
- Using multiple different perspectives facilitates the drawing out of explanatory links in different contexts (see, also, [Barton, 2017]).
- In particular, the phenomenon of convergence is an important for epistemic reasons.
- Insisting on one theory (possibly foundational) at the expense of others blinkers us here.

Conclusions

- We've seen in this talk that reasonable interpretations of the Logician's Dogma allow for easy Gettiering.
- The Logician's Dogma isn't the main target though: These types of Gettier cases indicate that shoring up possible explanatory links in mathematics is epistemically important.

Thanks! Discussion!

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