Large Cardinals and the Iterative Conception of Set

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Introduction

- When thinking about this presentation, I tried to tackle one of the most difficult questions facing philosophers (and academics more widely):

Problem.

How to provide a handout, when I'm always fiddling with my slides the night before?

- You can find these slides posted under the 'Blog' section of my website (https://neilbarton.net/). Just Google 'Neil Barton philosophy'.

Introduction

- Large cardinals are (rightly) viewed as some of the most interesting axioms of contemporary set theory.
- Indeed, the study of how they relate to consistency, model-building, and determinacy axioms has been one of the real great successes of the last century in set-theoretic mathematics.
- A lot of philosophical attention has been devoted to the justificatory case for large cardinals.
- The idea that the universe of sets should be maximal (or 'rich' or 'generous', or whatever) has sometimes been mobilised in their favour.

Plan

Target.

There are natural interpretations of maximality on which large cardinal axioms can fulfill most of their required foundational role, but are false.

- §1 What are large cardinals and why do we need them?
- §2 Maximality and large cardinals
- §3 Reinhardt cardinals and Choice
- §4 The Inner Model Hypothesis and Inaccessibles
- §5 Forcing Saturation and the Power Set Axiom
- §6 Strong absoluteness...speculative material warning!
- §7 The foundational role for large cardinals on these perspectives

§1 What are large cardinals and why do we need them?

- It's frustrating that there's no concrete formal definition of the notion of large cardinal axiom.
- However, they should have the feature that they transcend the consistency strength of previous large cardinals.
- This can be done by apparent brute size (e.g. inaccessible, hyper-inaccessible, Mahlo).
- Or through certain model-building properties (e.g. $0^\sharp$, the relationship between the least strong and least superstrong cardinal).

§1 What are large cardinals and why do we need them?

- In fact it turns out that the natural large cardinals are (at least apparently) linearly ordered by consistency strength.
- This gives us a first desirable use for large cardinals: Provide the indices of strength for any conceivable mathematics.

§1 What are large cardinals and why do we need them?

- However, large cardinals also find use in the construction of certain models.
- So, for example, we build models of the form $L[E]$ where L is an extender, using the large cardinal properties attaching to E to build the model.

§1 What are large cardinals and why do we need them?

- Closely linked to model building are the justificatory cases for axioms of definable determinacy.
- These axioms assert that certain games played with natural numbers have winning strategies, and have desirable consequences with respect to (for example) hereditarily countable sets.
- Importantly, they are implied by large cardinal axioms.
- For example, determinacy for projective sets is implied by the existence of infinitely many Woodin cardinals.
- One might then argue: If we justify the large cardinals, so we justify a nice theory (perhaps this even constitutes justification in itself, but I'll set this aside).

§2 Maximality and large cardinals

- Okay so, large cardinals are:
  1. Essential for indexing consistency strength.
  2. Useful for building models.
  3. Provide a possible justificatory case for axioms of definable determinacy.
- But why should we think that they are true?

§2 Maximality and large cardinals

- Here's one argument:(Before we begin, it is a bit of a straw man, so hold fire if you spot the problem! It will be useful later.)
- The iterative conception tells us to form all possible sets at each successor stage and continue that process as far as possible.
- Suppose we have some consistent large cardinal axiom.
- A large cardinal axiom asserts that the stages go as far as a certain ordinal.
- So if its consistent to go ahead and form a stage with a certain property, you should go ahead and do so.

§2 Maximality and large cardinals

- Some people (especially those who have experience with how large cardinal consistency and truth can relate) will have spotted why the argument is bad.
- You can have perfectly good models of set theory in which large cardinal consistency is not ratified by existence.
- This was (reportedly) Jensen's point concerning L.
- In this sense, we're not going to say anything new. But, it is argued that large cardinals represent principles that capture maximality.

§2 Maximality and large cardinals

"To answer this question [i.e. height maximality], a number of principles have been invoked. The ones that are probably best known are principles telling us, effectively, that the hierarchy goes at least as far as a certain ordinal. These include the Axiom of Infinity and the standard large cardinal axioms..." ([Incurvati, F], p.4)

"As with any large cardinal, positing a supercompact can be viewed as a way of assuring that the stages go on and on; for example, below any supercompact cardinal κ there are κ measurable cardinals, and below any measurable cardinal λ, there are λ inaccessible cardinals." ([Maddy, 2011] pp. 125–126)

- That's enough for now. But examples can be multiplied (e.g. [Hauser, 2001], parts of some textbooks e.g. [Drake, 1974]).

§3 Reinhardt cardinals and Choice

- Here's how we'll argue in the rest of the talk:
- The iterative conception legislates for forming all possible subsets at successor stages, and then iterating this as far as possible.
- So what if the subset forming operation at successor stages kills large cardinals?
- This isn't too challenging with examples like L, since that looks like a minimality principle (This has been discussed by Maddy—we'll see some discussion of Maddy's notion of restrictiveness later).
- But what if we can find maximality principles that kill large cardinals?

§3 Reinhardt cardinals and Choice

- Consider the Axiom of Choice.
- This is a maximality principle.
- If you've got a set x formed at $V_{\alpha+1}$, then you've got all members of x at latest at V_α, and so all elements of members of x at latest at V_α, and so a Choice set for x gets formed at latest at $V_{\alpha+1}$.
- (There are counterarguments here, but I don't think they pass muster, and you can bolster this argument in various ways (e.g. with second-order logic as in [Potter, 2004].)).

§3 Reinhardt cardinals and Choice

- But now we have the following:

Theorem.

[Kunen, 1971] Assuming **ZFC**, there are no Reinhardt cardinals.

- It's still open though whether there could be Reinhardt cardinals in a model satisfying only **ZF**.
- Moreover, there's an entire choiceless choiceless hierarchy with some nice consistency implications (see [Woodin, 2011] here), that would outstrip the usual hierarchy.
- Should we say Choice is false then?
- NO! The formation of Choice sets in V prohibits the formation of a stage with a Reinhardt cardinal.
- On the assumption that Reinhardts are consistent with **ZF** and realised in inner models, the action of the axiom asserting the existence of a Reinhardt cardinal is thus to minimise width.

§4 The Inner Model Hypothesis and Inaccessibles

- We can now make this phenomenon more extreme:

Definition.

[Friedman, 2006] The Inner Model Hypothesis states that if ϕ is true in an inner model I^{V^*} of an outer model V^* of V, then ϕ is true already in an inner model of V.

- In this way, the axiom asserts that V has been maximised with respect to internal consistency.
- Anything you can dream up in a class-like context, is already realised in a class-like context.
- Maybe this is a helpful way at getting at the idea of all possible subsets?
- There are some fiddly issues with coding here—see [Antos et al., S].

§4 The Inner Model Hypothesis and Inaccessibles

But we now have:

Theorem.

[Friedman, 2006] Suppose V satisfies the IMH. Then there are no inaccessibles in V.

However, we also have:

Theorem.

[Friedman et al., 2008] Suppose that V satisfies the IMH. Then V contains inner models with measurable cardinals (of arbitrarily large Mitchell order).

§4 The Inner Model Hypothesis and Inaccessibles

- We can actually make this slightly more problematic for the advocate of large cardinals.
- Consider Maddy's notion of a theory being restrictive:
- Roughly, a theory $\mathbf{T}_2$ maximises over another $\mathbf{T}_1$ iff $\mathbf{T}_2$ does one of:
  (I) Has an inner model with $\mathbf{T}_1$.
  (II) Has a truncation at an inaccessible with $\mathbf{T}_1$.
  (III) Has an inner model of a truncation at an inaccessible with $\mathbf{T}_1$.
- and has sets outside that interpretation.
- $\mathbf{T}_2$ properly maximises over $\mathbf{T}_1$, when one can't go back the other way.
- Usual example: $V = L$ vs. measurable cardinals.

§4 The Inner Model Hypothesis and Inaccessibles

Fact.

(see [Barton, S]) **NBG** + IMH properly maximises over **ZFC**+ "There are α-many measurables" for every α.

Fact.

If ϕ is a large cardinal axiom, **NBG** + IMH+ "There exists an inner model for ϕ" properly maximises over **ZFC** + ϕ".

- Maddy herself acknowledges that her notion isn't perfect, but it at least gives us a precise sense in which we might say that the Inner Model Hypothesis really does capture some maximising features.

§5 Forcing Saturation and the Power Set Axiom

- Okay, let's get X-treme.
- Forcing axioms are naturally understood as sorts of maximality principles, asserting that there are generics for certain kinds of well-behaved posets and families of dense sets.
- They can be understood as asserting that the universe has been saturated under forcing of a particular kind.
- In this way we might think of forcing as generating subsets given some subsets you already have.

§5 Forcing Saturation and the Power Set Axiom

So maybe we should have this axiom:

Definition.

We say that V satisfies the Forcing Saturation Axiom (or FSA) iff for any partial order $\mathbb{P} \in V$, and any family of dense sets $\mathcal{D} \in V$, there is a generic G for $\mathbb{P}$ and $\mathcal{D}$ in V.

- Of course, the FSA is inconsistent with **ZFC**!
- But maybe things are more subtle than that.
- Maybe there are so many subsets of an infinite set that they cannot all be collected at an additional stage.
- Maybe there is a different notion of collecting all possible subsets at successor stages.
- Maybe the Powerset Axiom is a kind of large cardinal axiom, that can only be true when we leave out subsets from the hierarchy.

§5 Forcing Saturation and the Power Set Axiom

- Okay, so drop the Powerset axiom, adopt the FSA, with the continuum now becoming a proper class...
- How do we define the iterative hierarchy here?
- Initial idea:

The Naive Forcing Saturated Hierarchy

is defined as follows (within **FSST**):

(I) $N_0 = \emptyset$

(II) $N_{\alpha+1} = Def(N_\alpha) \cup \{G | \exists \mathbb{P} \in N_\alpha \exists \mathcal{D} \in N_\alpha$ "$\mathbb{P}$ is a forcing poset $\mathcal{D}$ is a family of dense sets of $\mathbb{P}$ and G intersects every member of $\mathcal{D}$"$\}$

(III) $N_\lambda = \bigcup_{\beta<\lambda} N_\beta$

(IV) $N = \bigcup_{\alpha \in On} F_\alpha$.

Who can spot the problem?

§5 Forcing Saturation and the Power Set Axiom

- We need the generics to be fed in slowly and unboundedly.
- We can commit to a restricted form of possibility: You can only every grab at the 'next' generic in line.
- This is codified by a well-order R, and we have:

The Forcing Saturated Hierarchy

is defined as follows (within **FSST**):

(i) $F_0 = \emptyset$

(ii) $F_{\alpha+1} = Def(F_\alpha) \cup \{G | \exists \mathbb{P} \in F_\alpha \exists \mathcal{D} \in F_\alpha$ "$\mathbb{P}$ is a forcing poset $\mathcal{D}$ is a family of dense sets of $\mathbb{P}$ and G intersects every member of $\mathcal{D} \wedge G$ is the R-least generic for $\mathbb{P}$ and $\mathcal{D}$"$\}$

(iii) $F_\lambda = \bigcup_{\beta<\lambda} F_\beta$

(iv) $F = \bigcup_{\alpha \in On} F_\alpha$.

§5 Forcing Saturation and the Power Set Axiom

- F clearly satisfies **FSST**.
- It's not quite as neat as **ZFC** and the V_α, since there isn't a guarantee that F contains every set in a model of **FSST**. This is shown by the following:

Fact.

Over **ZFC**−Powerset, the FSA is equivalent to the claim "Every set is countable".

§5 Forcing Saturation and the Power Set Axiom

- However, we do have the following, if we modify Maddy's definition to consider theories extending **ZFC**$-$Powerset:

FACT.
Where ϕ is a large cardinal axiom, **FSST**$+$ "There is an inner model for **ZFC** $+ \phi$" properly maximises over **ZFC** $+ \phi$.

§6 Strong absoluteness

- This said, **FSST** on its own is weak (it can't even break $V = L$!) and we have to juice it up rather artificially.
- Are there natural axioms that imply that every set is countable, but would also maximise over standard **ZFC**-style set theories without artifice?
- Well, we are now a lot more free with what parameters we can have with our absoluteness principles:

Definition.

The Extreme Inner Model Hypothesis (or EIMH) states that if $\phi(\vec{a})$ is a formula containing arbitrary parameters $\vec{a} \in V$, then if $\phi(\vec{a})$ is true in an inner model of an outer model of V, then $\phi(\vec{a})$ is true in an inner model of V.

§6 Strong absoluteness

- The EIMH is somewhat strong: It implies the FSA, early indications are it breaks $V = L$, and it seems like one might be able to transfer techniques from the IMH to get some large cardinal strength.
- Unfortunately it's also too close for comfort to a well-known yet little-loved large cardinal axiom...$0 = 1$.

§6 Strong absoluteness

Theorem.(Probably.)

Let the Dependent Choice Scheme be the following scheme of assertions. For every second-order formula $\phi(X, Y, A)$ with class parameter A, if for every set X, there is a set Y witnessing $\phi(X, Y, A)$, then there is a single set Z making infinitely many dependent choices according to ϕ (i.e. $\forall X \exists Y \phi(X, Y, A) \rightarrow \exists Z \forall n \phi(Z_n, Z_{n+1}, A)$). Then:
If V satisfies $\mathbf{ZFC}^-$ with the Dependent Choice Scheme, it cannot tolerate the EIMH.

§6 Strong absoluteness

- We do have some models that violate the required DC principle (see work of Gitman and Friedman here).
- One immediate question then is if we can have the EIMH at all.
- But maybe there are versions of this that aren't so bad (e.g. just use ordinal parameters).
- There should be a whole space of hypotheses here...
- But both the mathematics and the philosophy needs to be worked out here—it's unclear what the space of positions looks like, and it's unclear how we might have an iterative picture.

§7 The foundational role for large cardinals on these perspectives

- What then of the foundational roles of large cardinals discussed earlier?
- Well, the indexing of consistency strength is unaffected.
- The case for determinacy is in principle unaffected, since the equivalence is actually with the existence of models, e.g.

Theorem.

TFAE:

1. Projective Determinacy (schematically rendered).
2. For every $n < \omega$, there is a fine-structural, countably iterable inner model $\mathfrak{M}$ such that $\mathfrak{M} \models$ "There are n Woodin cardinals".

- As it happens though, some of the principles we have considered do kill PD (e.g. IMH).

§7 The foundational role for large cardinals on these perspectives

- But it's at least open to hold that we may be convinced by PD on the basis of the various structural relationships exhibited in inner model theory, yet hold that large cardinals are killed.
- PD is, for example, perfectly compatible with **FSST**.
- We can also come up with IMH-like principles that kill some large cardinals but allow for PD (for example, just modify the IMH to only allow universes containing a proper class of Woodins).

§7 The foundational role for large cardinals on these perspectives

- For model building, the production of the canonical model is exactly linked to the determinacy axiom, not the large cardinal itself.
- So we can perfectly well have the construction of the canonical model without the literal truth of the large cardinal.
- Even if the determinacy does fail, given an inner model containing the large cardinal (which we often have for the theories discussed here), we can at least have a context in which the 'canonical' construction can be carried out.
- So whether you think these perspectives interfere with the foundational role for large cardinals, is somewhat dependent on exactly what you need/want.

Conclusion and an open question

- There's too many open questions regarding this material to list everything.
- We've seen some during the presentation.
- But I'm not here to suggest we should replace **ZFC**, or even standard large cardinals (though the questions raised merit answers).
- The main point is just that whether or not something actually counts as a maximality principle or not depends on prior commitments you may have about maximality.
- What we need then, is a careful disambiguation of the kind of maximality being employed.
- This is starting to get done to an extent (e.g. the unification of large cardinals under the philosophical idea of reflection, inner model hypotheses as absoluteness of the universe).
- But we need more real philosophical and mathematical labour here!

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