Large Cardinals and the Iterative Conception of Set: Is every set countable?

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18 January 2018

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- You can find these slides posted under the 'Blog' section of my website (https://neilbarton.net/).

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- Indeed, the study of how they relate to consistency, model-building, and determinacy axioms has been one of the real great successes of the last century in set-theoretic mathematics.
- A lot of philosophical attention has been devoted to the justificatory case for large cardinals.
- The idea that the universe of sets should be maximal (or 'rich' or 'generous', or whatever) has sometimes been mobilised in their favour.

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- §1 What are large cardinals and why do we need them?
- §2 Maximality and large cardinals
- §3 Reinhardt cardinals and Choice
- §4 The Inner Model Hypothesis and Inaccessibles
- §5 Forcing Saturation and the Power Set Axiom
- §6 Strong absoluteness...

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- §4 The Inner Model Hypothesis and Inaccessibles
- §5 Forcing Saturation and the Power Set Axiom
- §6 Strong absoluteness...speculative workshop material warning!
- §7 The foundational role for large cardinals on these perspectives

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- Or through certain model-building properties (e.g. $0^\sharp$, the relationship between the least strong and least superstrong cardinal).

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- This gives us a first desirable use for large cardinals: Provide the indices of strength for any conceivable mathematics.

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- For example, determinacy for projective sets is implied by the existence of infinitely many Woodin cardinals.
- One might then argue: If we justify the large cardinals, so we justify a nice theory (perhaps this even constitutes justification in itself, but I'll set this aside).

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- Okay so, large cardinals are:
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- But why should we think that they are true?

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- This was (reportedly) Jensen's point concerning L. (Since it's a workshop: Does anyone know where Jensen says this?)
- In this sense, we're not going to say anything new. But, it is argued that large cardinals represent principles that capture maximality.

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"As with any large cardinal, positing a supercompact can be viewed as a way of assuring that the stages go on and on; for example, below any supercompact cardinal κ there are κ measurable cardinals, and below any measurable cardinal λ, there are λ inaccessible cardinals." ([Maddy, 2011] pp. 125–126)

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- That's enough for now. But examples can be multiplied (e.g. [Hauser, 2001], parts of some textbooks (e.g. [Drake, 1974]).

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- But what if we can find maximality principles that kill large cardinals?

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- (There are counterarguments here, but I don't think they pass muster, and you can bolster this argument in various ways (e.g. with second-order logic as in [Potter, 2004].)).

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- Should we say Choice is false then?
- NO! The formation of Choice sets in V prohibits the formation of a stage with a Reinhardt cardinal.
- On the assumption that Reinhardts are consistent with **ZF** and realised in inner models, the action of the axiom asserting the existence of a Reinhardt cardinal is thus to minimise width.

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- Maybe this is a helpful way at getting at the idea of all possible subsets?

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[Friedman et al., 2008] Suppose that V satisfies the IMH. Then V contains inner models with measurable cardinals (of arbitrarily large Mitchell order).

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  (I) Has an inner model with $\mathbf{T}_1$.
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- and has sets outside that interpretation.
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- Usual example: $V = L$ vs. measurable cardinals.

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- Maddy herself acknowledges that her notion isn't perfect, but it at least gives us a precise sense in which we might say that the Inner Model Hypothesis really does capture some maximising features.

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- In this way we might think of forcing as generating subsets given some subsets you already have.

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- Maybe there is a different notion of collecting all possible subsets at successor stages.
- Maybe the Powerset Axiom is a kind of large cardinal axiom, that can only be true when we leave out subsets from the hierarchy.

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- How do we define the iterative hierarchy here?

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- Okay, so drop the Powerset axiom, adopt the FSA, with the continuum now becoming a proper class...
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The Naive Forcing Saturated Hierarchy

is defined as follows (within **FSST**):

(i) $N_0 = \emptyset$

(ii) $N_{\alpha+1} = Def(N_\alpha) \cup \{G | \exists \mathbb{P} \in N_\alpha \exists \mathcal{D} \in N_\alpha$ "$\mathbb{P}$ is a forcing poset $\mathcal{D}$ is a family of dense sets of $\mathbb{P}$ and G intersects every member of $\mathcal{D}$"$\}$

(iii) $N_\lambda = \bigcup_{\beta<\lambda} N_\beta$

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Who can spot the problem?

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- We need the generics to be fed in slowly and unboundedly.
- We can commit to a restricted form of possibility: You can only every grab at the 'next' generic in line.
- This is codified by a well-order R, and we have:

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- F clearly satisfies **FSST**.
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- However, we do have the following, if we modify Maddy's definition to consider theories extending **ZFC**−Powerset:

Fact.

Where ϕ is a large cardinal axiom, **FSST**+ "There is an inner model for **ZFC** $+ \phi$" properly maximises over **ZFC** $+ \phi$.

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- Are there natural axioms that imply that every set is countable, but would also maximise over standard **ZFC**-style set theories without artifice?
- Well, we are now a lot more free with what parameters we can have with our absoluteness principles:

Definition.

The Extreme Inner Model Hypothesis (or EIMH) states that if $\phi(\vec{a})$ is a formula containing arbitrary parameters $\vec{a} \in V$, then if $\phi(\vec{a})$ is true in an inner model of an outer model of V, then $\phi(\vec{a})$ is true in an inner model of V.

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- But maybe there are versions of this that aren't so bad (e.g. just use ordinal parameters).
- There should be a whole space of hypotheses here...
- But both the mathematics and the philosophy needs to be worked out here—it's unclear what the space of positions looks like, and it's unclear how we might have an iterative picture.

§7 The foundational role for large cardinals on these perspectives

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- What then of the foundational roles of large cardinals discussed earlier?

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- What then of the foundational roles of large cardinals discussed earlier?
- Well, the indexing of consistency strength is unaffected.
- The case for determinacy is in principle unaffected, since the equivalence is actually with the existence of models, e.g.

Theorem.

TFAE:

1. Projective Determinacy (schematically rendered).
2. For every $n < \omega$, there is a fine-structural, countably iterable inner model $\mathfrak{M}$ such that $\mathfrak{M} \models$ "There are n Woodin cardinals".

- As it happens though, some of the principles we have considered do kill PD (e.g. IMH).

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- But it's at least open to hold that we may be convinced by PD on the basis of the various structural relationships exhibited in inner model theory, yet hold that large cardinals are killed.
- PD is, for example, perfectly compatible with **FSST**.
- We can also come up with IMH-like principles that kill some large cardinals but allow for PD (for example, just modify the IMH to only allow universes containing a proper class of Woodins).

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- So we can perfectly well have the construction of the canonical model without the literal truth of the large cardinal.
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- So we can perfectly well have the construction of the canonical model without the literal truth of the large cardinal.
- Even if the determinacy does fail, given an inner model containing the large cardinal (which we often have for the theories discussed here), we can at least have a context in which the 'canonical' construction can be carried out.
- So whether you think these perspectives interfere with the foundational role for large cardinals, is somewhat dependent on exactly what you need/want.

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- This is starting to get done to an extent (e.g. the unification of large cardinals under the philosophical idea of reflection, inner model hypotheses as absoluteness of the universe).
- But we need more real philosophical labour here!

Thanks! Discussion!

Hugely grateful to:
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