My Research: Examining a mathematical world

Neil Barton*

25 August 2017

Introduction

My work centres on the *foundations* of mathematics. Immediately, one might ask "Foundations of mathematics? What on earth does that mean? I studied mathematics in highschool/college/university; we looked at numbers and geometry, and those were in perfectly good order. Why does mathematics *need* a foundation? Even if it did, what would such a foundation be *like*?".

These are all good questions in their own way, and many of them point to subtle and interesting topics in contemporary mathematical and philosophical research. However, as with many disciplines, the 'standard' mathematics with which one is familiar with in school is not the only game in town. Disciplines like number theory (the study of numbers like 0,1,2, and so on), analysis (the study of numbers you can represent with infinite decimals, and functions on them), and geometry (the study of abstract shapes and figures) are well-established and relatively uncontroversial. Just like in many of the sciences (e.g. theoretical physics) things get a bit more controversial when we get to the edges of mathematical research, however. Readers may be aware of strange theories on the boundaries of human understanding, like quantum mechanics or string theory. Just as these theories go well beyond what is taught in schools in terms of controversy and content, so the ever expanding frontier of mathematical research becomes more contestable. New mathematical structures are defined, and worries about whether the talk involving them really makes sense are often raised mathematicians and philosophers alike.

This one place where the foundations of mathematics comes in. Precisely what constitutes a foundation is a subject of much debate, but the core idea is to use well-understood notions to be precise about the principles we use when reasoning about mathematical objects. What kinds of mathematical structure are legitimate? What rules can we use when giving a mathematical proof? Are mathematical statements all either true or false? How is the study of esoteric mathematical objects related to more concrete applications of mathematical reasoning, such as in the sciences and computing? These are some of the questions researchers interested in the foundations of mathematics try and address.

If I've piqued your interest thus far, I'm happy. The study of mathematical foundations is rich and fascinating, and has given rise to some of the most incredible developments in human thought and technology.[1] You could stop here, but if you're curious, I've explained a little of my current and planned research below.

*Kurt Gödel Research Center for Mathematical Logic (KGRC), Währinger Straße, 25, 1090, Vienna, Austria.

[1]In fact, one of the central issues in the foundations of mathematics is an analysis of the notion of

1 Set theory

And central area of interest for me is the study of *sets* and mathematical notions of *infinity*. What is a set? Roughly speaking a set is a collection of other objects that we think of as an object in its own right. By analogy, one can think of *the* collection of people in a particular football team, or *the* bunch of grapes in the fruit bowl. Sets are a little different[2] from these kinds of collection, but for the purposes of giving a rough characterisation they can be thought of along these lines.

We talk about sets of mathematical objects too. I talk about the *set* of all natural numbers (which we denote by putting curly brackets around the objects as follows: $\{0, 1, 2, ..., n, ...\}$). We can then treat this as an object in its own right. Similarly, we could also consider the set of all rational numbers (i.e. all fractions):

$$\{1, 1/2, 2, 1/3, 3, 1/4, 2/3, 3/2, 4, ...\}$$

Alternatively we could talk about the set of all 'Real' numbers, which in addition the the rationals contains numbers like $\sqrt{2}$ and π. Set theory consists in the study of different mathematical systems describing sets.

The discipline is interesting both philosophically and mathematically. First, one can define any mathematical object through using sets. To do this, we first admit the empty set $\emptyset = \{\}$. This is a little puzzling, but its existence arises from most standard conceptions of set theory. We then (via some technical wizardry) define all the mathematical operations we know about (and many more) in terms of sets and their members. In this way, set theory gives us a good handle for when a new piece of mathematics is acceptable; if we can find a set-theoretic representation of some mathematical theory, then we know that it is at least as reliable as our talk about sets.

This ability to represent mathematical objects provides a second way in which set theory is very useful. By representing with sets, we can find representatives of our familiar mathematical objects in a given model (a model is just a bunch of mathematical objects and the relations between them). We can then study the relationships between them more easily; once a translation from a particular branch of mathematics has been given, we are able to prove new theorems about the represented objects by studying the properties of their set-theoretic representatives.

Third, and more philosophically, set theory provides us with our best theory of infinity. All the mathematical collections mentioned above are infinite. Things get weird though when we start comparing them, especially with respect to their *size*.

There are two main ways of comparing the 'size' of two sets x and y. One way is to examine whether or not every member of x is a member of y (then we say that x is a *subset* of y, written $x \subseteq y$). Then we know that x is at most the same size as y, and can be (in a sense smaller). Consider, for example, the even numbers $\{2, 4, 6, 8, ..., 2n, ...\}$ and the natural numbers $\{0, 1, 2, 3..., n, ...\}$. Now, since every member of $\{0, 2, 4, 6, 8, ..., 2n, ...\}$ is a member of $\{0, 1, 2, 3..., n, ...\}$ (and not vice versa) the set of all even numbers is (in a sense) 'smaller' than the set of all natural numbers.

However, what if we want to try and compare the sizes of two sets that don't share members? Which should we say is 'bigger' out of the set of all evens

computation. Before the many applications of these notions became apparent, many of these ideas were developed by different philosophers and logicians (as recently popularised by interest in the life and works of Alan Turing). Who knows what applications will be found for modern foundational techniques in the future?

[2]Specifically the identity of sets is determined by their members, whereas e.g. presumably the bunch of grapes can continue to be the same bunch if a eat just one of them.

$\{0, 2, 4, 6, ..., 2n, ...\}$ and the odd numbers $\{1, 3, 5, 7, ..., 2n + 1, ...\}$? No even number is odd, and vice versa, so how do we compare their size? We can do so by the notion of a *function* between the two sets. If we can match up every member of a set x with every single member of another set y, in such a way that no two members of y are assigned the same member of x, then we can 'pair up' all the elements of x with elements of y. In this sense, then, the two sets would have the same 'size'. In this *new* sense of size (known as 'cardinality') the set of odd numbers and even numbers have the same size, the function $f(x) = x + 1$ does exactly this job. Visually, you can think of this as follows:

$$\begin{array}{ccccc} 0 & 2 & 4 & 6 & ... \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ 1 & 3 & 5 & 7 & ... \end{array}$$

You can see that the function pairs up a single even number with a single odd number, and we say they have the same cardinality.

However, under this new notion of size, the set of all even numbers has the same size as all the natural numbers. The function $f(x) = 2x$ does the job nicely:

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & ... \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & 2 & 4 & 6 & ... \end{array}$$

So, there's a pairing of the even numbers with the natural numbers. Things get wackier still; despite the fact that there appear to be infinitely many rational numbers between 0 and 1, we can nonetheless define a function of the required kind between the set of all natural numbers and the set of all rationals.

You might then think that there's just one size of infinity. Not so; it turns out that some infinite sets are 'bigger' than others in this sense of cardinality. In particular, the real numbers are 'bigger' than the rationals. This theorem can be made even more general; in 1891 Georg Cantor showed that for any set x, the set of all its subsets is 'bigger' than x. This means that, if you accept that for any set there is a set of all its subsets (a standard principle in many set theories) there are, in fact, unboundedly many different 'sizes' of infinite set. The world of infinite sets appears to be a pretty diverse place, and set theory provides us with our most rigorous way of reasoning with these strange entities.

1.1 How many universes?

Okay, good. It looks like there are decent reasons to be examine set theory; it's of interest as a theory in its own right and also facilitates the study of other mathematical objects. However, things get exciting when you see what sorts of questions our standard set theory can and cannot answer.

Here's a really natural question to ask once you realise that, for any set x, the set of all subsets of x is bigger than x (this was Cantor's Theorem from above):

> **Question.** For any set x, are there any sets bigger than x, but smaller than the set of all its subsets? In other words, given a set, are there intermediate cardinalities between its cardinality and the cardinality of the set of all its subsets?

This question (known as the Generalised Continuum Hypothesis) turns out to not be answerable on the basis of our standard set-theoretic rules that we lay down. More precisely, you can show[3] that there are structures which satisfy the above question in the affirmative, and others for which there are no intermediate cardinalities between a set and the set of all its subsets.

There are several interesting features of this result. First, it is just bizarre that our standard set-theoretic principles, which we might think of as telling us what sets exist, fail to yield answers to this sort of fundamental and natural question. Second, the manner in which the results are proved is important. What we do, is take a particular set-theoretic structure, and start adding sets to it in a clever way (a technique known as *forcing*). This allows us to change the answer to the above question by adding sets to structures (if you add subsets of a set, you can introduce intermediate cardinalities, and if you add in functions pairing up sets with your original set x, you can destroy these intermediate cardinalities). Importantly, all these structures look very 'normal' (in ways that can be made mathematically precise[4]) from a set-theoretic point of view.

This inability to settle questions on the basis of our standard theories is not just confined to the Generalised Continuum Hypothesis. There is a vast array of techniques of different kinds, that allow us to move between natural looking structures that disagree on the truth value of various sentences. This has lead some people to question whether or not there really is just one domain of sets. Given that we prove these results by adding sets to a particular structure, what is to say that there is just one structure? There are now two competing views: the 'Universe' view holds that there is just one maximal domain of sets, while 'Multiverse' views hold that there is not one 'absolute' universe but rather many domains which we move between using mathematical constructions. Each position recommends its own philosophical and mathematical research projects. Throughout my research, I've examined how adopting one or the other viewpoint affects one's conception of mathematics, and ways in which we might incorporate insights from one view into the other.

1.2 Searching for new axioms

A central question, once one realises that statements like the Generalised Continuum Hypothesis cannot be resolved on our standard conception of sets, is whether or not we can justify new principles to resolve such 'independence'. Throughout human history, when faced with a seemingly unsolvable problem, human ingenuity has triumphed and we have found a solution. The study of mathematics has been no different; new theories and structures are being found all the time that shed additional light on our intellectual endeavours. Given then that there are some sentences that are neither provable nor refutable under our most widely accepted theory of sets, can we justify new axioms to resolve this independence?

The idea of justifying new axioms is an interesting proposal. Central to my research in this field is an analysis of how we conceive of mathematical reality, and what the theorems and proofs we currently know about might tell us about its nature. The hope then is to either justify new axioms, thereby enabling us to settle previously unanswered questions (and ask new ones!), or explain why no such project of justification can ever be successful.

[3] This was done by [Gödel, 1940] and [Cohen, 1963].

[4] For those that know; the structures remain transitive and well-founded.

2 Pluralism in foundations

In the previous section, I mentioned that part of my research centred on incorporating the insights from one view into another. We might push this further, and argue that rather than trying to settle the debate, we *should* encourage researchers working in both fields. That way, the understanding gained from the study of one view can be brought to bear on the other (and vice versa), and if one programme ends up falling flat, then we will still have enough diversity in our knowledge to keep pressing forward.

One idea here is to develop views which *reconcile* competing pictures. By aiming to dissolve the apparent tension between different theories, we could operate with each, as long as we bear in mind how they should be understood. Part of my research aims at developing a deeper understanding of mathematics that will allow us to reconcile multiverse and universe viewpoints. However, there are even more directions in which we might push this thought:

2.1 What things?

Set theory isn't the only foundational theory on the market. There are other foundational languages in which we can interpret mathematical discourse. A good example here is *category theory* which takes the notion of a *mapping* as primitive, rather than a *collection* (as in set theory). Under category theory, we think of mathematical operations that take you between objects. For example, in set theory, we might represent the numbers $0, 1, 2, 3$ as the sets:

$$0 = \emptyset, 1 = \{\emptyset\}, 2 = \{\emptyset, \{\emptyset\}\}, 3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}.$$

and more generally, $n = \{0, 1, 2, 3, .., n - 1\}$

In *category theory*, we instead think of the numbers $0, 1, 2$ as the maps, *add* 0, *add* 1, and *add* 2. Interestingly, category theory turns out to be very good for representing our *structural* thinking. For example, the two sets $\mathbb{Z}^+ = \{1, 2, 3, ..., n, ...\}$ and $\mathbb{Z}^- = \{-1, -2, -3, ..., -n, ...\}$ in a sense have *the same structure* despite being different things. This is a feature of mathematical reasoning that category theory captures succinctly. In particular, in set theory we *build* the objects up from objects we already have. However, in category theory we just state what external mapping relationships an object must have in order to fulfill a desired role.

Historically, category theory and set theory have been regarded as competing disciplines, with fierce arguments on both sides. More recently, however, mathematicians and philosophers have begun to examine the ways in which they interact. Part of my ongoing research is to examine framework theories in which the two foundations can be fruitfully related.

2.2 What logic?

If I say:

> "It is not the case that it is not raining"

does that come down to the same thing as saying that it is raining (and vice versa)? There's a school of thought (called *intuitionism*), opposing classical logic (which does accept this), that says it might not. More generally, intuitionists do not accept the classical rules that "not not P" always implies "P" or that "P or not P" holds for any P.

This especially goes in the case of mathematics. Given that we can't decide some statements on the basis of particular mathematical principles, we might think that they are, in fact, neither true nor false (and hence "P or not P" doesn't hold). Moreover, there are some cases where thinking in classical terms is unhelpful for reasoning; things just sometimes work more neatly in the intuitionist framework.[5] Part of my work extends to analysing when we should be using intuitionist logic, and what this tells us about the ordinary practice of classical mathematics.

3 My 'philosophy' of philosophy

If you've got this far, congratulations and thanks! It's always nice to have people interested in your research, so please do get in contact if you have any questions. To finish, I just wanted to point to a final theme that ran throughout this statement of research. By and large, I didn't put forward many views as unequivocally *true*. This isn't to say that I always think there's no fact of the matter about the questions I address, but it does indicate how I prefer to practice philosophy.

For me, philosophy is about finding out how different theoretical presuppositions combine to form whole worldviews. I'm therefore far less interested in pushing my own particular favourite framework, and much more keen to see what views we can fit together and how. For this reason, most of my projects focus on ways of reconciling previously thought to be incompatible viewpoints, and bringing the tools of one to bear on the other. My experience has been that everyone has something insightful to say, and we learn best by trying to understand each other and synthesising our ideas. While we should continue to resist each other and subject our views to rigorous scrutiny, this should be construed as a creative process for developing more robust theories. It is through this competitive collaboration that we can best increase our understanding of the world.

References

[Cohen, 1963] Cohen, P. (1963). The independence of the continuum hypothesis. *Proceedings of the National Academy of Sciences of the United States of America*, 50:1143–1148.

[Gödel, 1940] Gödel, K. (1940). *The Consistency of The Continuum Hypothesis*. Princeton University Press.

[5]A good example here is the study of *infinitesimals*; numbers bigger than zero but smaller than any positive real number. They are certainly very strange (if they exist at all), but seem to correspond very neatly to human thinking in calculus.